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Covariant action for conformal higher spin gravity

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Abstract

Conformal higher spin (HS) gravity is a HS extension of Weyl gravity and is a family of local HS theories, which was put forward by Segal and Tseytlin. We propose a manifestly covariant and coordinate-independent action for these theories. The result is based on an interplay between HS symmetries and deformation quantization: a locally equivalent but manifestly background-independent reformulation, known as the parent system, of the off-shell multiplet of conformal HS fields (Fradkin–Tseytlin fields) can be interpreted in terms of Fedosov deformation quantization of the underlying cotangent bundle. This brings into the game the invariant quantum trace, induced by the Feigin–Felder–Shoikhet cocycle of Weyl algebra, which extends Segal's action into a gauge invariant and globally well-defined action functional on the space of configurations of the parent system. The same action can be understood within the worldline approach as a correlation function in the topological quantum mechanics on the circle.

Keywords: conformal higher spin gravity, Hochschild cocycle, deformation quantization

(Some figures may appear in colour only in the online journal)

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Contents

1. Introduction

There are very few examples of (covariant) actions for higher spin (HS) gravities at present. (1) In three dimensions, there is a class of topological HS theories [1–8], which encompasses massless, partially-massless and conformal (HS) fields. The actions for these theories are simply the Chern–Simons action for various Lie algebras that can be thought of as the HS extensions of Poincaré, (anti-)de Sitter or conformal algebras. (2) In the light-cone gauge, Chiral HS gravity in flat space admits a very simple action [9–13]. The theory has two contractions [14], which can be understood as HS extensions of self-dual Yang–Mills and of self-dual gravity theories, and have simple covariant actions [15]. Some recent progress has been made towards uplifting Chiral theory to twistor space [16–19], where the Chern–Simons action [17] captures correctly the cubic interactions both in flat and (A)dS spaces. (3) IKKT model for a HS algebra [20–22] is an example of a non-commutative field theory with HS fields in the classical limit. (4) The last but not the least is the class of conformal higher spin (CHS) gravities [23–25], which is the subject of this paper.

Conformal higher spin gravities (CHS gravity) are HS extensions of conformal gravity. In four dimensions, it is an extension of Weyl gravity. More generally, conformal gravities are theories of a metric $g_{\mu\nu}(x)$ that are invariant under diffeomorphisms and local Weyl rescalings

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = \Omega^2(x)\, g_{\mu\nu}(x)\,. \tag{1.1}$$

They exist in even dimensions $n \geqslant 4$ and the cases of $n=2$ and $n=3$ are somewhat special, the latter one admitting a Chern–Simons formulation [26, 27], which was discussed above. In $n=4$ dimensions there is a unique action, the Weyl action, while for $n>4$ there is an ambiguity growing with dimension in what conformal gravity is, which is related to the growing number of conformal invariants (see e.g. [28–30] for explicit expressions in 6 and 8 dimensions, and also [31] for the recent progress concerning the general classification).

It is remarkable that diffeomorphisms and Weyl rescalings can be extended to an infinite-dimensional algebra of symmetries that acts on an infinite multiplet of CHS fields [24, 32], which are also known as Fradkin–Tseytlin fields [33]. The multiplet contains fields with arbitrarily high spin. CHS gravities are theories for this multiplet and there seems to be only one such theory for every multiplet in even dimensions $n=2p$, $p \geqslant 2$. Two seemingly different constructions were proposed: Tseytlin's that is based on the effective action approach [23] (see also [25]); Segal's that is tight to a worldline model and deformation quantization [24]. Both of these approaches are eventually closely related to each other and prove the existence of CHS gravities.

Let us comment on possible relevance of CHS gravity in applications, starting with its spin-2. Weyl gravity represents one of the contributions to the conformal anomaly, i.e. to an anomalous piece of the effective action of a conformal field theory in the gravitational background. It also appears as an anomalous contribution to the holographic effective action [34]. Similar ideas can be applied to CHS gravity [23, 35]. At the same time Weyl gravity action gives an example of a conformal invariant while its equations of motion give an example of a conformally invariant differential operator, which are notoriously difficult to construct and which are of substantial interest in conformal field theory and conformal geometry. Likewise, CHS gravity provides a HS extension of these operators, see e.g. [36, 37]. Another potentially interesting application of CHS gravity is that massless HS gravity could be recovered from CHS one just like Einstein gravity with negative cosmological constant can be identified as a subsector of the Weyl gravity [38].

One aspect of CHS gravity and, more generally, of 'higher spin geometry' we would like to improve on is to propose a manifestly covariant and both coordinate- and background-independent construction for CHS gravities, including the action principle. Having such a formulation is important for any extension of gravity and should facilitate the study of these theories in the future, e.g. propagation of CHS fields on gravitational backgrounds [39–45].

The problem of covariantization of CHS gravity leads to, roughly speaking, two problems: (i) what is the proper HS analog of the covariant derivative ∇? (ii) what is the proper HS analog of $\sqrt{g}$ to be able to integrate? None of these objects makes sense *a priori* when HS gauge fields are present. Indeed, HS gauge transformations, of which the spin-two symmetries (diffeomorphisms and Weyl rescalings in our case) form a small subset, mix spins and derivatives and, hence, neither of ∇ and $\sqrt{g}$ transform in a meaningful way. In a broader sense, the real question is 'what is higher spin geometry?'.

A key step to the solution was given already in [24]. Spin-two (low-spin) symmetries such as diffeomorphisms, Weyl and Yang–Mills symmetries, can be represented by operators of the first order at most (in spacetime derivatives). Higher spin transformations bring in higher derivatives. Therefore, the natural language is that of differential operators. The latter can also be represented locally as the Moyal–Weyl star-product algebra. The action of CHS gravity is then a specific invariant functional on the Moyal–Weyl algebra, which takes advantage of the invariant trace $\mathrm{Tr}(\bullet)$.

Moyal–Weyl star-product is defined in terms of Darboux coordinates and does not support diffeomorphism symmetry[6]. It known how to fix this problem within the framework of deformation quantization—one needs to resort to the Fedosov approach [46], which is based on picking a background connection. From the gauge field theory perspective, resorting to Fedosov quantization amounts to the so-called parent reformulation of the system. In the case of the Segal system, the corresponding parent reformulation is known [47] (see also [48, 49]; strictly speaking we employ a certain partially gauge-fixed version of this formulation) and its equations of motion are those of the Fedosov-like quantization of the corresponding constrained system defined on the cotangent bundle of the space-time manifold. The crucial point of this parent system is that it is background-independent because the Fedosov-like connection becomes a genuine gauge field and hence no background fields are needed in the construction. As a result, we get an off-shell gauge theory that contains the CHS multiplet together with all the necessary auxiliary and pure gauge fields that encode derivatives thereof in a HS covariant way.

The last ingredient is Feigin–Felder–Shoikhet cocycle [50] that allows one to define the invariant trace $\mathrm{Tr}(\bullet)$ on the Fedosov-quantized symplectic manifold. This cocycle is a by-product of Shoikhet–Tsygan–Kontsevich formality [51, 52]. As a result, Segal's action, i.e. CHS gravity action in Darboux coordinates, is a particular gauge of our action and one can choose other gauges that make various aspects manifest, e.g. one can pursue metric-like or frame-like approaches.

It was shown in [53] that the Feigin–Felder–Shoikhet cocycle can be represented as a specific correlator in a topological quantum mechanics of a free particle on the circle. At the same time, Segal's action also admits a worldline formulation [24, 54, 55]. Therefore, it should not be surprising that our covariant CHS gravity action can be represented as a correlation function in a worldline model.

The outline of the paper is as follows. In section 2, we introduce CHS fields, Fradkin–Tseytlin fields, and review the two approaches to CHS gravities. In section 3, we introduce the parent extension of the Segal off-shell system and present our main result—a covariant action for CHS gravities. In section 4, we discuss possible gauge conditions which allow us to reformulate the action in terms of the independent and unconstrained fields. We also elaborate on the explicit relation between frame-like and metric-like formulations of the system, which are derived from the parent formulation through suitable gauge conditions. We end up with some conclusions and a discussion.

2. CHS gravity

We begin by outlining the problem of CHS gravity and a closely related issue of a HS extension of conformal geometry. Next, we review two approaches, Tseytlin's and Segal's, to CHS gravity. At the end, we discuss the relation between the two. While each of the approaches gives a 'proof of concept' for the existence of theory, we will see that obtaining a covariant and globally well-defined form of it may not be straightforward, and will propose a solution to this problem in section 3.

[6] In the sense that its very definition on a symplectic manifold relies on a choice of coordinates—called Darboux coordinates—wherein the symplectic form has constant components, and hence the star-product is defined only locally.

2.1. CHS fields

As previously stated, CHS gravity is an extension of conformal, or Weyl, gravity, whose spectrum contains fields of all integer spins. One way to describe Weyl gravity is in terms of an equivalence class of conformal metrics, i.e. a rank-2 symmetric tensor $g^{\mu\nu}$, subject to the gauge transformations

$$\delta_{\xi,\sigma} g^{\mu\nu} = \mathcal{L}_\xi\, g^{\mu\nu} + 2\sigma g^{\mu\nu}\,, \tag{2.1}$$

where $\xi \equiv \xi^\mu(x)\,\partial_\mu$ is a vector field that generates infinitesimal diffeomorphism and $\sigma(x)$ is an arbitrary function parameterizing infinitesimal Weyl rescalings (1.1).

CHS fields in the metric-like approach [33] are a natural generalization of the above gauge symmetry to totally-symmetric tensors of arbitrary ranks $s > 2$. More precisely, CHS fields can be described by symmetric tensors $\Phi^{\mu_1\cdots\mu_s}$ with $s \in \mathbb{N}$. The theory turns out to be uniquely fixed by its (HS) non-abelian gauge symmetries whose exact form can be read off from a simple matter coupling and is reviewed in section 2.3. To envisage the theory, it can be helpful to look for natural gauge symmetries for $\Phi^{\mu_1\cdots\mu_s}$. To this effect, it is important to isolate the spin-two field $g^{\mu\nu}$ and treat it as a background. A HS extension of (2.1) can be looked for, starting from

$$\delta_{\xi,\sigma}\Phi^{\mu_1\cdots\mu_s} = \mathcal{L}_\xi\,\Phi^{\mu_1\cdots\mu_s} + w_s\,\sigma\,\Phi^{\mu_1\cdots\mu_s} + \nabla^{(\mu_1}\xi^{\mu_2\cdots\mu_s)} + g^{(\mu_1\mu_2}\sigma^{\mu_3\cdots\mu_s)} + \cdots\,. \tag{2.2}$$

Here, the first term, the Lie derivative, declares $\Phi^{\mu_1\cdots\mu_s}$ to transform as a tensor under diffeomorphisms and the second term assigns a certain Weyl weight w_s to it; the third term is a generalization of diffeomorphisms to HSs (at this point the spin-two background is important and we use $\mathcal{L}_\xi\, g^{\mu\nu} = \nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu$ to propose a HS extension); the fourth term represents a HS Weyl transformation, but again over the spin-two background; the dots ... at the end are meant as a reminder that this expression needs to be completed with terms that are nonlinear in HS fields themselves.

Upon linearization around flat space, the diffeomorphisms get reduced to the Poincaré symmetries and the HS gauge transformations become[7]

$$\delta_{\xi,\sigma}\Phi_{a_1\ldots a_s} = \partial_{(a_1}\xi_{a_2\ldots a_s)} + \eta_{(a_1a_2}\sigma_{a_3\ldots a_s)}\,, \tag{2.3}$$

where η_{ab} is the flat metric[8]. A free action for such fields is simply given by[9]

$$S_s[\Phi] = \int d^n x\, \Phi^{a_1\ldots a_s}\, P^{b_1\ldots b_s}_{a_1\ldots a_s}(\partial)\,\Box^{s+\frac{n-4}{2}}\,\Phi_{b_1\ldots b_s}\,, \tag{2.4}$$

where $P^{b_1\ldots b_s}_{a_1\ldots a_s}(\partial)$ is a traceless and transverse projector, thereby ensuring the invariance of this action under the linear gauge symmetries (2.3). For example, for $s = 1$ we have $P^b_a = \delta^b_a - \partial_a\partial^b/\Box$. These projectors were initially derived in four dimensions in [63, 64] and in arbitrary dimensions in [24], and have found applications in e.g. [25, 55, 65, 66]. The right power of $\Box$ in (2.4) is to ensure locality of the action.

Upon integrating by part, this action can be brought to the form

$$S_s[\Phi] = (-1)^s \int d^n x\, C^{a_1\ldots a_s,b_1\ldots b_s}\,\Box^{\frac{n-4}{2}}\, C_{a_1\ldots a_s,b_1\ldots b_s}\,, \tag{2.5}$$

[7] This type of conformal gauge fields was first introduced in $4d$ [56] as sources to traceless conserved tensors.

[8] We use Greek indices $\mu,\nu,\ldots$ to denote (co)tangent indices on a (generally) curved manifold and Latin indices $a,b,\ldots$ refer to flat space or fiber indices.

[9] Generalizations to supersymmetric cases [33, 42, 44, 45, 57–61] and mixed-symmetry fields [62] are also possible, but will be studied elsewhere. In the present paper, we deal with the most basic example of the bosonic CHS gravity with only totally symmetric fields in the spectrum.

where

$$C_{a_1\dots a_s,b_1\dots b_s} := \partial_{a_1}\dots\partial_{a_s}\Phi_{b_1\dots b_s} + \text{permutations} - \text{traces}, \tag{2.6}$$

is the (linearized) spin-s Weyl tensor, where the permutations and traces in the above formula implement the projection onto the traceless rectangular Young diagram of length s. The tracelessness of the spin-s Weyl tensor ensures its invariance under the spin-s Weyl transformations. For spin-2, this reproduces the usual linearized Weyl tensor and hence the linearized action of Weyl gravity. For a proof of the conformal invariance of the linearized action, see [24, section 10].

One can proceed starting from the free actions (2.4) or (2.5) and look for cubic and higher interaction vertices while deforming the free gauge symmetry (2.3) accordingly, which is often called Noether procedure (or gauging in supergravity). While this approach should, in principle, allow one to address the problem of constructing CHS gravity in a systematic way, it is notoriously difficult in practice, see [43–45, 57, 58, 67] for some results in this direction.

In general, the introduction of interactions is strongly constrained by the requirement of Weyl invariance, and in particular, a simple dimensional argument shows that all possible vertices compatible with conformal symmetry and involving a finite number of fields will contain a finite number of derivatives, and hence the resulting theory will be local. Indeed, within the perturbative expansion over flat space the conformal dimension of $\Phi_{a_1\dots a_s}$ is $(2-s)$ and an interaction vertex for k fields with spins s_i schematically reads

$$S_k \sim \int d^n x\,(\partial)^p\,\Phi^{(s_1)}\dots\Phi^{(s_k)}\,. \tag{2.7}$$

Therefore, it has to have a fixed number of derivatives $p = n + \sum_i (s_i - 2)$. Unfortunately, the price to pay for locality is that CHS gravity is not unitary (as could be expected from the fact that its kinetic terms are of higher derivative type), which is true already for its spin-two subsector, Weyl gravity. Nevertheless, it is an interesting example of HS gravity that has applications within AdS/CFT duality [34, 68–74] and conformal geometry, since it produces many conformally invariant operators. The S-matrix has good chances to be 1 due to the HS symmetry [75, 76] and, hence, the theory may turn out to be unitary/trivial, which is a sign of integrability.

Leaving aside the question of classifying possible CHS gravities, there are two concise recipes to construct specific examples of such theories that we are going to review now. These are Tseytlin's approach [23] that is based on the idea of effective action and Segal's approach [24] that draws inspiration from studying the quantized particle model coupled to background HS fields.

2.2. Tseytlin's approach: induced action

The approach due to Tseytlin [23] (see also [25] for a more elaborated exposition) rests on the idea of induced actions, already discussed by Sakharov [77] for gravity. It consists in considering a (complex) conformal scalar field ϕ with (HS) currents $J_{a_1\dots a_s}$ coupled to a background of CHS fields, i.e.

$$S_h[\phi] = \int d^n x \left(\phi^*\,\Box\phi + \sum_{s=0}^{\infty} J_{a_1\dots a_s} h^{a_1\dots a_s}\right) = \int d^n x\,\phi^*\left(\Box + \widehat{H}\right)\phi\,. \tag{2.8}$$

Here, $J_{a_1\dots a_s}$ are bilinear in the scalar field ϕ and read schematically

$$J_{a_1\dots a_s} = \phi^*\partial_{a_1}\dots\partial_{a_s}\phi + \dots\,, \tag{2.9}$$

where ... complete it to a traceless (can be achieved off-shell) and conserved (on-shell) tensor. The fields $h^{a_1\cdots a_s}$ can be treated as sources for these currents or, and this is an interpretation we need, as background fields. In the last step, we took the liberty to integrate by parts so that no derivatives act on ϕ^* and all the derivatives involved in the definition of the currents are distributed over ϕ and $h^{a_1\cdots a_s}$. As a result of this resummation, $\widehat{H}$ is the (higher order) differential operator acting on ϕ

$$\widehat{H} = \sum_k \widehat{H}^{a_1\cdots a_k}(h)\,\partial_{a_1}\dots\partial_{a_s}\,, \tag{2.10}$$

whose coefficients depend on the original sources $h^{a_1\cdots a_s}$. Let us note for the future that $\widehat{H} = \widehat{H}(x,\partial)$ is, basically, a generic formally Hermitian[10] differential operator. Since the HS currents are conserved and traceless the background fields $h^{a_1\cdots a_s}$ enjoy the same symmetry as (2.3). It is important to stress that the simple Noether coupling (2.8) is not off-shell gauge invariant under (2.3) and, as usual, requires higher order corrections to the gauge symmetry and to the Noether coupling (so-called Seagull terms). In the simplest low spin cases, the complete coupling for the spin-one background field A_μ originates from $D_\mu\phi^* D^\mu\phi$ with $D_\mu = \partial_\mu + iA_\mu$ and for the spin-two background, i.e. the conformal metric itself, the stress-tensor coupling $T_{ab}h^{ab}$ has to be appended with infinitely many terms that can be resummed into the action of the conformally coupled scalar field

$$S[\phi] = \int d^n x\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + \tfrac{(n-2)}{4(n-1)}\phi^* R\phi\right). \tag{2.11}$$

Suppose we are given an action $S_h[\phi]$ for the conformal scalar field ϕ coupled to an arbitrary CHS background (collectively denoted by h). One can then compute the effective action

$$e^{-W[h]} = \int \mathcal{D}\phi^*\mathcal{D}\phi\, e^{-S_h[\phi]}\,, \tag{2.12}$$

using the heat kernel method

$$W[h] = -\int_\epsilon^\infty \tfrac{dt}{t}\,\mathrm{Tr}\, e^{-t\widehat{F}}\,, \qquad \widehat{F} := \Box + \widehat{H}, \tag{2.13}$$

where ε is a cut-off. Using the heat kernel expansion, the effective action can be written as

$$W[h] = (\text{poles in } \epsilon) + a_{n/2}[\widehat{F}]\,\log\epsilon + (\text{series in } \epsilon)\,, \tag{2.14}$$

where the Seeley–DeWitt coefficient $a_{n/2}[\widehat{F}]$ of the operator $\widehat{F} = \widehat{F}[h]$ appears only in even dimension n. The pole part represents the usual UV divergences. The coefficient $a_{n/2}$ is a local functional of the CHS fields h and is Weyl invariant (see e.g. [25, section 3] for more details and [55] for a recent discussion). Let us stress that we restrict ourselves to field configurations where only finite number of component fields from h are nonzero and hence the order of $\widehat{H}$ is finite as well (for more details on the heat kernel of higher-order operators see e.g. [78, 79]).

The Seeley–DeWitt coefficient $a_{n/2}[\widehat{F}]$ in (2.14) has all the desired properties that one would ask of an action for CHS gravity. In other words, the CHS gravity action in Tseytlin's approach is defined as the logarithmically divergent piece of the effective action of a scalar field conformally coupled to a background of CHS fields. It is also well-known that for the low spin

[10] By 'formally' Hermitian here we simply mean that $F^\dagger = F$, where $f(x)^\dagger = f^*(x)$, $\partial_a^\dagger = -\partial_a$ and $\dagger$ is an anti-involution, $(FG)^\dagger = G^\dagger F^\dagger$.

background fields A_μ and $g^{\mu\nu}$ in, say $n=4$, we have [41, section 3]

$$a_{(n=4)/2} = \frac{1}{(4\pi)^2}\int d^4x\,\sqrt{g}\left(-\tfrac{1}{12}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{120}C_{\mu\nu,\lambda\rho}C^{\mu\nu,\lambda\rho}\right), \tag{2.15}$$

where $C_{\mu\nu,\lambda\rho}$ is the Weyl tensor, and the topological Euler term was dropped. Therefore, it should not be too surprising that $a_{n/2}$ receives well-defined, local HS corrections as long as we switch on the HS background fields. One subtlety can be that $\widehat{F}$ is a higher derivative operator and the heat kernel techniques for general higher derivative operators are yet to be developed (see e.g. [80, 81] for recent works on heat kernels for higher-order operators). The induced action approach should be generalizable to mixed-symmetry and supersymmetric cases. In particular, it has recently been advanced to $\mathcal{N}=1$ supersymmetric CHS case [45, 61] and the quadratic part of the action has been obtained via the induced action technique [61].

2.3. *Segal's approach: particle in a HS background*

Suppose we are given the quantized phase-space, i.e. an associative algebra of functions on the phase space $\mathbb{R}^{2n}$ with linear coordinates x^μ, p_ν, subject to $\{x^\mu, p_\nu\} = \delta^\mu_\nu$. Here and in what follows we employ the language of Weyl symbols, i.e. identify the operator algebra of polynomial functions p with coefficients in smooth functions in x tensored with $\mathbb{R}[[\hbar]]$ (formal power series in $\hbar$) and with the product being Moyal–Weyl star-product,

$$(f\star g)(x,p) = f(x,p)\,\exp\left[\frac{\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p_\mu}\frac{\overrightarrow{\partial}}{\partial x^\mu}\right)\right] g(x,p)\,, \tag{2.16}$$

so that in particular, the commutation relations between the coordinates x^μ and p_ν read

$$[x^\mu, p_\nu]_\star = \hbar\,\delta^\mu_\nu\,. \tag{2.17}$$

This algebra admits an anti-involution defined by

$$x^\dagger = x\,, \qquad p^\dagger = p\,, \qquad \hbar^\dagger = -\hbar\,, \qquad (a\star b)^\dagger = b^\dagger \star a^\dagger\,. \tag{2.18}$$

We also employ the standard representation of the algebra on functions in x^μ tensored with $\mathbb{R}[[\hbar]]$. More precisely, any $f(x,p)$ is sent to a differential operator $\widehat{f}(x,\frac{\partial}{\partial x})$ that is obtained by employing the symmetric ordering. This gives a quantization map. Its inverse, sending operators to phase space functions is usually referred to as the symbol map. Under the quantization map, the involution $\dagger$ is sent to the formal adjoint with respect to the standard inner product

$$\langle\phi,\psi\rangle = \int d^nx\,\phi^*\,\psi\,. \tag{2.19}$$

The above construction is invariant under general linear transformation of x complemented by the conjugate transformations of p. In particular, 'wave-functions' ϕ, ψ are assumed to be semi-densities for the inner product to be invariant under such transformations. Note that more general diffeomorphisms cannot be naturally implemented in this framework since, for instance, they do not preserve Moyal–Weyl star-product (2.16).

Let $F(x,p)$ be a generic element of the algebra which we view as a first class constraint (we do not explicitly indicate the $\hbar$ dependency, having in mind that we work over $\mathbb{R}[[\hbar]]$). Assuming F Hermitian and polynomial in p_a, one can easily write the free action of the associated scalar field as

$$S[\phi,F] = \langle\phi,\widehat{F}\phi\rangle\,. \tag{2.20}$$

In this action, we view F as a generating function for background fields, and $\phi=\phi(x)$ as a complex scalar field. Indeed, the relation to the previous section is that in the action (2.20) we can absorb $\Box$ into $\widehat{F} = \Box + \widehat{H}$, i.e. $\eta^{\mu\nu} p_\mu p_\nu$ is a (spin-two) background value for F and the perturbations correspond to turning on a CHS background. In particular, it is more natural to associate the Taylor coefficients of F when expanded in p

$$F = \sum_s F^{\mu_1\cdots\mu_s}(x)\, p_{\mu_1}\dots p_{\mu_s}\,, \tag{2.21}$$

with the background fields (recall that they can be obtained via a certain rearrangement of $h^{a_1\cdots a_s}$).

The advantage of the action (2.20) is that it is very easy to determine the full gauge symmetry that leaves it invariant. Indeed, it is obvious that it is invariant under the following infinitesimal gauge symmetries,

$$\delta_{\xi,w} F = \frac{1}{\hbar}\,[F,\xi]_\star + \{F,w\}_\star\,, \tag{2.22a}$$

$$\delta_{\xi,w}\phi = -\left(\tfrac{1}{\hbar}\,\widehat{\xi} + \widehat{w}\right)\phi\,, \tag{2.22b}$$

where the gauge parameters ξ and w are Weyl symbols of the Hermitian operators: $\widehat{\xi}^\dagger = \widehat{\xi}$, $\widehat{w}^\dagger = \widehat{w}$ and where 'hat' denotes the quantization map. These symmetries have a natural interpretation in terms of the Hamiltonian constrained system describing the underlying particle model, see appendix A for details. An even more compact way to represent the same gauge symmetry is to define $u = \hbar^{-1}\xi + w$, which is neither Hermitian nor anti-Hermitian, and write

$$\delta_u F = u^\dagger \star F + F \star u\,, \qquad\qquad \delta_u \phi = -\widehat{u}\,\phi\,. \tag{2.23}$$

In what follows we refer to the off-shell system (2.22*a*) with field F subject to the above gauge symmetries as the *off-shell Segal system*. In other words, we drop the ϕ-part. The off-shell Segal system defines a completion of (2.2) and also solves the problem raised in the previous section, of how to couple matter fields to a HS background.

Let us comment on the background independence of the system. At first glance, the definition of the system does not involve any background fields and hence the system can be considered as a gravity-type theory (for the moment defined at the off-shell level only). Nevertheless, a more careful analysis shows that the star-product involved in the construction depends on a choice of Darboux coordinates. Although this background dependence is not of a fundamental nature and can, in principle, be excluded by working with differential operators as such, it is not clear if this does not lead to further complications. More geometrically, what we are implicitly dealing with is a cotangent bundle over the spacetime manifold $\mathcal{X}$ and it is known that one, at least, needs to fix an affine connection on $\mathcal{X}$ to define a concrete star-product in a coordinate invariant way. As we are going to see in the next section, this drawback can be cured by passing to a locally equivalent reformulation of the system.

By analyzing the component form of the transformation (2.22*a*), one finds that they generalize gauge transformations of the conformal gravity and, at the same time, give a nonlinear extension of the linearized gauge symmetries of the Fradkin–Tseytlin fields. Note however that to see that the linearization of the above gauge symmetries indeed reproduces the Fradkin–Tseytlin gauge transformations, one is to employ an intricate field redefinition, introduced in [24] (see also [25, 47]). Let us give a few sketchy arguments supporting these statements. First of all, we note that the above gauge transformations contain a subalgebra of diffeomorphisms and Weyl rescalings which act on the spin-2 component of F in a standard way, but at the same time affects other components of F. More precisely, consider transformations with the parameters

$$\xi = \xi^\mu(x) p_\mu \,, \qquad\qquad w = \sigma(x) \,. \tag{2.24}$$

The commutator of two such transformations, with $u_i = \frac{1}{\hbar}\xi_i^\mu(x)p_\mu + \sigma_i(x)$ for $i = 1,2$, reads as

$$[u_1, u_2]_\star = \tfrac{1}{\hbar}\,(\xi_2^\mu \partial_\mu \xi_1^\nu - \xi_1^\mu \partial_\mu \xi_2^\nu) p_\nu + \xi_2^\mu \partial_\mu \sigma_1 - \xi_1^\mu \partial_\mu \sigma_2 \,, \tag{2.25}$$

and hence this subalgebra is isomorphic to the semidirect product of the algebra of diffeomorphisms (represented by vector fields) with the (abelian) algebra of Weyl rescalings. However, this subalgebra is not represented in a canonical way on the p-Taylor coefficients of F as one can see this already with the first even-spin components of the field F

$$F = D(x) + \tfrac{1}{2}\, g^{\mu\nu}(x)\, p_\mu p_\nu + \cdots \,, \tag{2.26}$$

where notation $g^{\mu\nu}$ will justify itself immediately. Here we assume that the HS components vanish otherwise there can be higher derivative contributions below. The subalgebra (2.24) of the gauge transformations, which consists of diffeomorphisms and Weyl rescalings, acts as

$$\delta_{\xi,\sigma} g^{\mu\nu} = -\partial_\lambda \xi^\mu g^{\lambda\nu} - \partial_\lambda \xi^\nu g^{\lambda\mu} + \xi^\lambda \partial_\lambda g^{\mu\nu} + 2\sigma g^{\mu\nu} \equiv \mathcal{L}_\xi\, g^{\mu\nu} + 2\sigma g^{\mu\nu} \,, \tag{2.27}$$

$$\delta_{\xi,\sigma} D = \xi^\mu \partial_\mu D + 2\sigma D - \frac{1}{2}\hbar^2 \partial_\mu \partial_\nu \xi^\lambda \partial_\lambda g^{\mu\nu} - \frac{1}{2}\hbar^2 g^{\mu\nu} \partial_\mu \partial_\nu \sigma \,. \tag{2.28}$$

We see that $g^{\mu\nu}$ transforms as a conformal metric should, but D has additional non-covariant terms (the last two terms, proportional to $\hbar^2$) that get D entangled with $g^{\mu\nu}$. In principle, there is a field-redefinition $D \to D + Y(g)$ that allows one to eliminate such terms [24]. However, it is, generally, difficult to find one and it becomes more and more complicated to disentangle transformations as we proceed to HSs. While some formulas, e.g. (2.25) and (2.27), indicate that the diffeomorphism algebra is a part of the off-shell Segal system, others such as (2.16) and (2.28) make it very difficult to see how it is realized on fields[11]. This is one of the problems that the present paper is to solve: how to covariantize the Segal approach[12].

Genuine HS gauge transformations (2.3) can be seen once we expand (2.22) over the background $F^{(0)} = \frac{1}{2}p^2 \equiv \frac{1}{2}\eta^{\mu\nu} p_\mu p_\nu$. Indeed, for $F = \frac{1}{2}p^2 + f$, the linearized gauge transformations read

$$\delta_{\xi,w} f = \tfrac{1}{2\hbar}\,[p^2, \xi]_\star + \tfrac{1}{2}\,\{p^2, w\}_\star \,, \tag{2.29}$$

and for generic ξ and σ we find

$$\delta_{\xi,w} f(x;p) = -p^\mu \partial_\mu \xi(x;p) + p^2\, \sigma(x;p) + \ldots \,, \tag{2.30}$$

where $\ldots$ denotes corrections of order $\mathcal{O}(\hbar^2)$ which are also responsible for mixing the fields of different spins and for the non-covariance. When Taylor expanded in p the above formula reproduces (2.3) to the leading order in $\hbar$. This way ξ and w represent *higher spin (HS) diffeomorphisms* and *higher spin (HS) Weyl transformations*.

To summarize, the algebra of diffeomorphisms and Weyl rescalings is obviously a part of the gauge symmetries of the off-shell Segal system. In addition, the linearization over $\frac{1}{2}p^2$ does reproduce the HS transformations (2.3). Therefore, altogether we are in a possession of a

[11] Retrospectively, this is consistent with the fact that we are dealing with symbols of differential operator, for which it is known that only the principal part (coefficients of its term of highest order in derivatives) transforms as a totally symmetric tensor (see e.g. [82] for more details on this approach in relation to higher spin gravity).

[12] For the moment we only talk about the off-shell Segal system, whose covariantization is known [40, 47].

certain completion of (2.2). Nevertheless, the diffeomorphisms and Weyl transformations are not represented in the canonical way and the Taylor coefficients of F do not behave as tensors under diffeomorphisms.

The off-shell Segal system can be expanded over any background, the simplest one being $\frac{1}{2}p^2$, which is also the one making (2.20) into the action of a conformal scalar field.

Given a fixed background solution, a natural question is to consider an algebra of its symmetries. By definition these are the leftovers of gauge symmetries that leave it intact:

$$u^\dagger \star p^2 + p^2 \star u = 0\,. \tag{2.31}$$

We also note that the symmetries (2.23) are reducible: $\delta_u F = 0$ for $u = i\,v \star F$ with $v^\dagger = v$. Therefore, the symmetry algebra of the vacuum $\frac{1}{2}p^2$ is exactly the algebra of 'higher symmetries of Laplacian' [24, 83–85], i.e. the quotient algebra of the symmetries modulo trivial ones. At the same time, this algebra is the deformation quantization [85–87] of the coadjoint orbit corresponding to the free scalar field in flat space, $\Box\phi = 0$, as a representation of the conformal algebra $\mathfrak{so}(n,2)$. This is an infinite-dimensional associative algebra $\mathfrak{hs}(\Box\phi)$ that can also be obtained as a quotient of $\mathcal{U}(\mathfrak{so}(n,2))$ by a certain two-sided ideal, known as the Joseph ideal [88] (see also [89, 90] and references therein for further applications to the construction of various higher algebras).

A simple generalization of this construction is to take the same scalar matter $\phi(x)$, but with a different kinetic term, $\Box^k\phi$, which are called higher order singletons [71]. Obviously, they correspond to $(p^2)^k$ within the Segal construction. The corresponding HS algebras $\mathfrak{hs}(\Box^k\phi)$ are not isomorphic for different values of the integer k [86, 91–95], the theories have different spectra of HS currents and, hence, the background fields they couple to. Therefore, one can have an access to a larger set of CHS gravities by picking different vacua. It is a vital point of Segal's construction that even though most of the discussion does not have to refer to any particular vacuum, it is necessary to make a choice at some point since different vacua correspond to different spectra of CHS fields, i.e. to essentially different theories.

As we discussed above, the action for the background fields can be obtained as a log-divergent piece in the effective action of a conformal scalar field coupled to the HS background. Here, we recall that within the approach of Segal the action is defined as

$$S[F] = \int d^n x\, \mathcal{L}_x(F)\,, \qquad\qquad \mathcal{L}_x(F) = \int d^n p\, \mathcal{L}_{x,p}(F)\,, \tag{2.32}$$

where $\mathcal{L}_x(F)$ is the Lagrangian that can be proved to be a local function of the p-Taylor components of F and x-derivatives thereof. The Lagrangian $\mathcal{L}_x(F)$ results from evaluating an integral over p for an auxiliary Lagrangian density $\mathcal{L}_{x,p}(F)$ on the phase-space. In other words,

$$S[F] = \mathrm{tr}\,\mathcal{L}_{x,p}(F) = \int d^n x\, d^n p\, \mathcal{L}_{x,p}(F)\,, \tag{2.33}$$

where tr denotes the usual trace in the Weyl quantization given by the phase-space integral of the respective symbol. The phase-space Lagrangian $\mathcal{L}_{x,p}(F)$ is a specific star-product function satisfying

$$\mathcal{L}'_{x,p}(F) \star F = 0 = F \star \mathcal{L}'_{x,p}(F)\,, \tag{2.34}$$

for any function F. It is easy to see that this property indeed guarantees gauge invariance of (2.32). At the formal level this $\mathcal{L}_{x,p}$ function can be defined as the star-product Heaviside step-function,

$$\mathcal{L}_{x,p}(F) = \Theta_\star(F)\,. \tag{2.35}$$

While some care is needed to work with such an object, see [24] and appendix B, it can be shown to be well-defined[13].

Let us sketch the proof of the gauge invariance of the action. There are two types of gauge transformations involved in (2.22): (a) adjoint transformations via commutator $[F,\xi]_\star$ and (HS) Weyl transformations via anti-commutator $\{F,w\}_\star$. The invariance under (a) is manifest due to the use of star-product and of the invariant trace tr, i.e. its cyclic property $\mathrm{tr}[f,g]_\star = 0$. The invariance under (b) is what fixes $\mathcal{L}_{x,p}$ to be the star-product Heaviside function, since it requires

$$\delta_w S = \mathrm{tr}\,\delta_w \mathcal{L}_{x,p}(F) = \mathrm{tr}\,\left(\mathcal{L}'_{x,p}(F)\star\{F,w\}_\star\right) = 2\mathrm{tr}\,\left(\mathcal{L}'_{x,p}(F)\star F\star w\right) = 0\,,$$

where we used the cyclicity of the trace. Similarly to $\Theta'(x)x = \delta(x)x \equiv 0$, the last equality formally implies $\mathcal{L}_{x,p}(F) = \Theta_\star(F) + \mathrm{const}$. The constant does not give any contribution in the Segal case, but it does lead to an 'index' for the covariantized CHS gravity we discuss in section 3.

2.4. Tseytlin/Segal dictionary

At first sight, it may seem that Tseytlin's and Segal's constructions are completely unrelated. Indeed, computationally this is true to an extent: in Tseytlin's approach one has to extract the log-divergent pieces of one-loop Feynman diagrams of matter fields with HS background fields on external lines; in Segal's approach one is to expand the star-product Heaviside step-function. Nevertheless, one can give an argument [24, section 6.2] (see also [96, appendix D]) for why the end result has to be the same.

Let us start from Tseytlin's definition,

$$S[\widehat{F}] = a_{\frac{n}{2}}[\widehat{F}]\,, \tag{2.36}$$

and re-write it in terms of the heat kernel for $\widehat{F}$. The latter admits a small t expansion [25],

$$\mathrm{Tr}\big(e^{-t\widehat{F}}\big) = t^{-\frac{n}{2}}\sum_{k=0}^{\infty} t^k\, a_k[\widehat{F}]\,, \tag{2.37}$$

so that the CHS gravity action reads

$$S[\widehat{F}] = \tfrac{1}{2i\pi}\oint \tfrac{dt}{t}\,\mathrm{Tr}(e^{-t\widehat{F}})\,, \tag{2.38}$$

where the integral is over a closed contour including the origin of the complex plane. Now recall that, given two differential operators $\widehat{D}_1$ and $\widehat{D}_2$, with symbols D_1 and D_2 respectively, the symbol of their composition $\widehat{D}_1\circ\widehat{D}_2$ is the star product of their symbols, $D_1\star D_2$. As a consequence, the symbol of $e^{-t\widehat{F}}$ is given by the star-exponential of the symbol F of $\widehat{F}$, i.e.

$$e_\star^{-tF} := \sum_{k=0}^{\infty}\tfrac{(-t)^k}{k!}F^{\star k}\,,\qquad F^{\star k} := \underbrace{F\star\cdots\star F}_{k\,\text{times}}\,. \tag{2.39}$$

On top of that, the trace of a differential operator agrees with the trace of its symbol,

$$\mathrm{Tr}(\widehat{D}) = \mathrm{tr}(D) \tag{2.40}$$

[13] For example, there is also a sort of regularization involved into the definition of $\Theta_\star(F)$ and one of the crucial steps is to pick the vacuum $F^{(0)} = \frac{1}{2}p^2$ and treat all the other fields as small perturbations (large perturbations can reach a non-equivalent vacuum). Also, evaluation of the p-integral requires Euclidean signature, but once $\mathcal{L}_x(F)$ is computed, the corresponding action is gauge invariant for any choice of the signature.

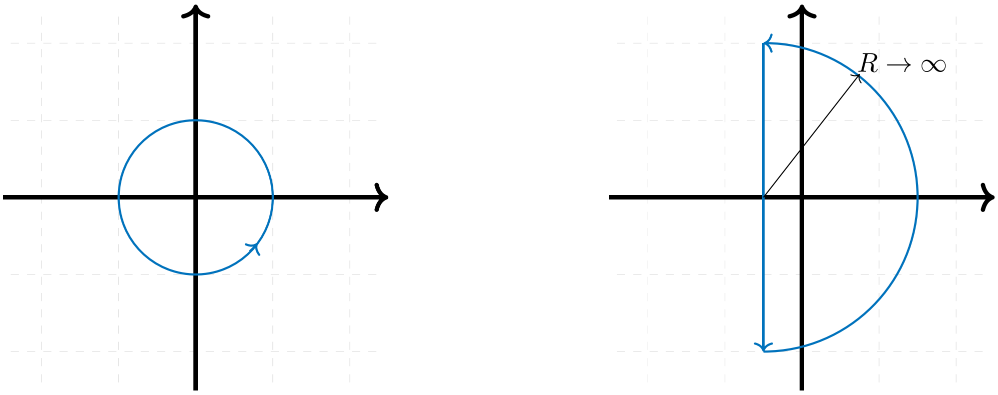

Figure 1. Deformation of the contour used to pick up the Seeley–DeWitt coefficient of the logarithmic divergence in the trace of the heat kernel.

so that the CHS gravity action can be written as

$$S[\widehat{F}] = \tfrac{1}{2i\pi} \oint \tfrac{dt}{t}\, \mathrm{tr}(e_\star^{-tF})\,. \tag{2.41}$$

Assuming that the trace of the star-exponential of $(-tF)$ is analytic in the $\Re(t) > 0$ region of the complex t-plane, we can deform the original closed contour around the origin to a half-circle whose diameter consists of the line defined by $\Re(t) = -\epsilon$ with $\epsilon \to 0^+$ (see figure 1). Further assuming that $\mathrm{tr}(e_\star^{-tF})$ goes to 0 when $|t| \to \infty$, the contribution of the contour integral on the arc of the half-circle vanishes as the radius is sent to infinity, so that the action (2.41) now reads

$$S[\widehat{F}] = \lim_{\epsilon\to 0^+} \tfrac{1}{2i\pi} \int_{-\infty}^{+\infty} \tfrac{d\tau}{\tau - i\epsilon}\, \mathrm{tr}(e_\star^{i\tau F})\,. \tag{2.42}$$

Finally, assuming that we can exchange the order of the trace and the contour integral, the above action can be re-written as

$$S[\widehat{F}] = \mathrm{tr}\big(\Theta_\star(F)\big)\,, \tag{2.43}$$

where

$$\Theta_\star(a) := \lim_{\epsilon\to 0^+} \tfrac{1}{2i\pi} \int_{-\infty}^{+\infty} \tfrac{d\tau}{\tau - i\epsilon}\, e_\star^{i\tau a}\,, \tag{2.44}$$

is the Heaviside star-function, as introduced by Segal [24]. Even though some of the steps followed previously may seem a bit formal (in the sense that we had to assume some analytical properties of $e_\star^{-tF}$ and its trace), Segal showed how to compute the quadratic and cubic pieces of the above action with $F = \frac{1}{2}p^2 + \mathcal{O}(h)$, where h denote CHS fields (see appendix B for more details).

3. Covariant action for CHS gravity

As it is clear from the review of the two approaches to CHS gravity recalled in the previous section, there is no doubt that this class of theories does exist and comprises well-defined local field theories. CHS gravity is, first of all, a theory of gravity and, hence, it is important to have the diffeomorphism/Weyl symmetry and HS extensions thereof represented in a simple way. This is the goal of the present section.

3.1. Parent form of the off-shell Segal system

The off-shell Segal system (2.22) can be equivalently reformulated in a coordinate-independent and globally well-defined way with the help of a version of the Fedosov quantization approach. From the field theory perspective, this amounts to what is known as the parent reformulation [47–49]. Its application to CHS gravity was already detailed in [40, appendix A].

The reformulation is constructed as follows. We consider a formal version of the quantized phase space (2.17). This is obtained by taking Weyl algebra $\mathcal{A}_{2n}$ of polynomials in p_a with coefficients in formal power series in y^a with the product being the Moyal–Weyl star-product, denoted $*$ so as to distinguish it from the previous section (as before the algebra is considered as that over $\mathbb{R}[[\hbar]]$). Given an n-dimensional space-time manifold $\mathcal{X}$, consider a Weyl algebra bundle $W(\mathcal{X})$ over $\mathcal{X}$, whose fiber at $x \in \mathcal{X}$ is a Weyl algebra $\mathcal{A}_{2n}$, associated with $T_x\mathcal{X} \oplus T^*_x\mathcal{X}$. In other words, sections of this bundle are elements of $\Gamma\big(S(T\mathcal{X})\big) \otimes \Gamma\big(\widehat{S}(T^*\mathcal{X})\big)$ where S denote the symmetric algebra and $\widehat{S}$ its completion[14]. Consider then a $\mathcal{A}_{2n}$-valued connection 1-form A and a section F of $W(\mathcal{X})$, subject to the following equations[15]:

$$dA + \tfrac{1}{2\hbar}[A,A]_* = 0\,, \qquad\qquad dF + \tfrac{1}{\hbar}[A,F]_* = 0\,, \tag{3.1}$$

and the following gauge symmetries

$$\delta_\xi A = d\xi + \tfrac{1}{\hbar}[A,\xi]_*\,, \qquad \delta_{\xi,w} F = \tfrac{1}{\hbar}[F,\xi]_* + \{F,w\}_*\,, \tag{3.2}$$

where ξ and w are sections of $W(\mathcal{X})$, i.e. locally they are zero-forms with values in $\mathcal{A}_{2n}$, and w satisfies

$$dw + \tfrac{1}{\hbar}[A,w]_* = 0\,. \tag{3.3}$$

Note that the gauge parameter w is subject to a differential constraint. In fact, it is equivalent to an algebraic one provided that the term $dx^\mu e^a_\mu(x) p_a$ in A, which is linear in p and y-independent, is determined by an invertible $e^a_\mu(x)$, as we assume below. Indeed, this condition then allows one to uniquely express all the components of w in terms of $w|_{y=0}$, as is explained in section 4. We also assume that $g^{ab}(x)$ determining the term quadratic in p and y-independent, $\frac{1}{2}g^{ab}(x)p_ap_b$, in F is invertible.

The above system is obtained as a partial gauge fixing of the equivalent (locally in the space-time) representation of the off-shell Segal system proposed in [47]. In particular, because the relevant gauge transformation is algebraic and the gauge condition is strict, the gauge fixing produces an equivalent system. Moreover, the above system is also off-shell. The system (3.1)–(3.3) will be called *parent Segal system* and solves the problem of covariantizing the off-shell Segal system (2.22). Note also that if one omits the gauge transformations with parameters w, equations (3.1) and (3.2) are precisely the defining equations for the Fedosov-like connection and the lift of functions in the version of the Fedosov quantization suitable for cotangent

[14] This distinction between the symmetric algebra of $T\mathcal{X}$ and the completion of the symmetric algebra of $T^*\mathcal{X}$ reflects the fact that we are considering polynomials in p (coordinates on the fibers of $T^*\mathcal{X}$) and formal power series in y (coordinates on the fibers of $T^*\mathcal{X}$)..

[15] Note that a simple extension of this parent system would be to require that the curvature of A takes values in the center of the Weyl algebra (i.e. is simply given by a closed 2-form) instead of vanishing, exactly like in Fedosov quantization.

bundles. These equations were also discussed in [47, 97] as an off-shell system for massless HS fields.

It is easy to give an independent proof of the equivalence by employing a special gauge where $A = dx^\mu \delta^a_\mu p_a$[16]. As such, this gauge is reachable locally, see section 4 and appendix D for details. We will also call it Segal's gauge, for it is easy to make contact with the Segal approach of section 2.3. Indeed, in this gauge

$$F = F_0(x+y,p)\,, \qquad \xi = \xi_0(x+y,p)\,, \qquad w = w_0(x+y,p)\,, \tag{3.4}$$

and the residual gauge symmetry reproduces the off-shell Segal system in terms of the initial data F_0, ξ_0 and w_0 (which are the y-independent piece of these 0-forms). Note that ξ is unconstrained in (3.2), but the requirement to maintain Segal's gauge implies $d\xi + \frac{1}{\hbar}[A,\xi]_* = 0$, i.e. the parameters of the residual transformations are covariantly constant.

Although equations (3.1) and (3.3) involve space-time derivatives, they are equivalent to algebraic equations as they allow one to reconstruct a solution from the initial data at $y = 0$. Indeed, as it is easy to observe in Segal's gauge, the initial data for F, ξ and w is given by arbitrary functions of x and p. The fiber-wise Moyal–Weyl star-product in $y-p$ space induces the one on x and p, and we can recover all formulas from section 2.3.

The notation suggests that ξ is responsible for a covariantized version of (HS) diffeomorphisms and w is responsible for (HS) Weyl transformations. A special feature of this formulation is that the actual Segal gauge transformations (2.22) are associated with covariantly constant ξ and w. Parameters w of (HS) Weyl symmetry are always constrained by (3.3) in order for the transformed F to obey (3.1). Therefore, Segal gauge transformations should be associated with

$$\delta_\xi A = 0\,, \qquad \delta_{\xi,w} F = \tfrac{1}{\hbar}[F,\xi]_* + \{F,w\}_*\,, \tag{3.5}$$

where ξ and w obey

$$d\xi + \tfrac{1}{\hbar}[A,\xi]_* = 0\,, \qquad dw + \tfrac{1}{\hbar}[A,w]_* = 0\,. \tag{3.6}$$

From the point of view of physical fields hidden in A and F (as the coefficients of their power series expansion in y and p), gauge transformations with unconstrained parameters ξ represent field redefinitions, which we discuss later in section 4. Further reduction of symmetries is possible if we consider those that preserve a given background $A = A^{(0)}$ and $F = F^{(0)}$. This way we recover the HS algebra $\mathfrak{hs}(\Box\phi)$ for $F^{(0)} = \frac{1}{2}p^2 \equiv \frac{1}{2}\eta^{ab}p_a p_b$ and $A^{(0)} = dx^\mu \delta^a_\mu p_a$.

The advantage of the parent reformulation is that it is globally well-defined for any space-time manifold $\mathcal{X}$[17]. In particular, it is manifestly coordinate independent and does not require any predefined background geometrical structures. In order to illustrate this property, let us mention that the system encodes a particular star product on $T^*\mathcal{X}$, which is still determined by an affine connection on $\mathcal{X}$ (along with extra structures, in general) hidden in the field A, but now A is a part of the field content and not a predefined background field.

The parent Segal system allows us to parameterize covariant derivatives of the physical fields hidden in the initial data $F|_{y=0}$ and $A|_{y=0}$ as auxiliary fields inside F and A that are reconstructed by solving (3.1). Since the parent Segal system gauges the HS diffeomorphisms ξ of the off-shell Segal system, (3.1) encodes fully HS covariant derivatives of the initial data,

[16] In particular, this implies picking a coordinate frame where $e^a_\mu = \delta^a_\mu$. In the Fedosov deformation quantization of general symplectic manifolds the analogous gauge can be chosen locally and requires Darboux coordinates on the base manifold.

[17] Of course, the existence of Lorentzian metric imposes some restrictions on the topology of $\mathcal{X}$.

i.e. roughly speaking, it defines for us a HS covariant derivative $d + \frac{1}{\hbar}[A, \bullet]$. In order to construct an action, we will also need an appropriate HS covariant version of the measure $\sqrt{g}$, which is related to the quantum trace discussed below.

3.2. Covariant action

What we are after is an invariant definition of Segal system, which does not depend on the particular choice of coordinates and/or predefined geometric structures such as an affine connection determining the star-product. The first step is to reformulate the system in the parent form (3.1)–(3.3). In the second step, we define an action in terms of the parent fields using a version of the invariant trace proposed by Feigin *et al* [50, section 4]. More specifically, their construction allows one to define a trace over the algebra of functions of a symplectic manifold endowed with a star-product obtained using Fedosov quantization, in an invariant way. In the case of the flat symplectic manifold $\mathbb{R}^{2n}$, the trace reduces to the usual integral of the phase space, used in Segal's formulation of CHS gravity.

The core of the construction is the Hochschild $2n$-cocycle with values in $\mathcal{A}_{2n}^*$,

$$\Phi : \underbrace{\mathcal{A}_{2n} \otimes \ldots \otimes \mathcal{A}_{2n}}_{2n+1\,\text{times}} \longrightarrow \mathbb{C} \tag{3.7}$$

of the Weyl algebra $\mathcal{A}_{2n}$ generated by formal power series in y^a and polynomials in p_a, see appendix C for further details. Using this cocycle, one can build a reduced polylinear map

$$\mu : \mathcal{A}_{2n}^{\otimes(n+1)} \longrightarrow \mathbb{C}[p] \,, \tag{3.8}$$

defined as[18]

$$\mu(a_0|a_1, \ldots, a_n) = [\Phi](T_{p'}a_0; T_{p'}a_1, \ldots, T_{p'}a_n, y^{b_1}, \ldots, y^{b_n})\epsilon_{b_1\ldots b_n}|_{p'_a = p_a} \,, \tag{3.9}$$

where $T_{p'}a(y,p) = a(y,p+p')$ and $[\Phi]$ denotes the antisymmetrization of Φ in its $2n$ arguments, i.e. $[\Phi]$ is the associated Chevalley–Eilenberg $2n$-cocycle with values in $\mathcal{A}_{2n}^*$.

Let us list here the algebraic properties of the map μ:

(*i*) Total antisymmetry in its n last arguments,

$$\mu(a_0|a_{\sigma_1}, \ldots, a_{\sigma_n}) = (-1)^{|\sigma|}\,\mu(a_0|a_1, \ldots, a_n) \,, \tag{3.10}$$

for any elements $a_0, a_1, \ldots, a_n \in \mathcal{A}_{2n}$ of the Weyl algebra and any permutation $\sigma \in \mathcal{S}_n$;

(*ii*) The normalization condition,

$$\mu(f; p_{a_1} \ldots, p_{a_n}) = \tfrac{1}{n!}\,\epsilon_{a_1\ldots a_n} f|_{y=0} \,, \tag{3.11}$$

for any $f \in \mathcal{A}_{2n}$;

(*iii*) The 'cocycle condition', modulo total derivative in p-space,

[18] Another option is to plug in $dp_a y^a$ in the last slots and get a cocycle with values in top-forms of p-fiber which is a natural integration object over fiber (see also appendix D).

$$\sum_{i=0}^{n}(-1)^i\mu([a_{-1},a_i]_*;a_0,\dots,\widehat{a}_i,\dots,a_n)$$
$$+\sum_{0\leqslant i<j\leqslant n}(-1)^{i+j}\mu(a_{-1};[a_i,a_j]_*,a_0,\dots,\widehat{a}_i,\dots,\widehat{a}_j,\dots,a_n)=\tfrac{\partial}{\partial p_a}\varphi_a(a_{-1};a_0,\dots,a_n)\,, \tag{3.12}$$

for some $\varphi_a(a_{-1}|a_0,\dots,a_n)\in\mathbb{C}[p]$ (see appendix C for more details);

(*iv*) The $\mathfrak{sp}(2n,\mathbb{R})$-invariance,

$$\mu(a;-,\dots,-)=0=\mu(-;-,\dots,a,\dots,-)\,, \tag{3.13}$$

for any element $a\in\mathfrak{sp}(2n,\mathbb{R})\subset\mathcal{A}_{2n}$[19].

The above structure allows us to define an action principle as follows. Let $l_*(F)$ be a star-product function (see appendix B for more details), and consider the following functional

$$S[A,F]=\int_{\mathcal{X}}\int_{p-\text{fiber}}\mu\big(l_*(F);A,\dots,A\big)\,, \tag{3.14}$$

where the fields A and F are subject to the off-shell constraints (3.1). This action is invariant under the gauge transformations generated by ξ, up to boundary terms: indeed, upon using the property (*iii*) of the map μ, together with the flatness and covariant constancy of A and F respectively, one can show that (see corollary C.3 in appendix C)

$$\delta_\xi\mu\big(l_*(F);A,\dots,A\big)\propto d(\dots)+\tfrac{\partial}{\partial p_a}(\dots)_a\,, \tag{3.15}$$

i.e. the integrand of (3.14) is gauge invariant up to a total derivative. Note that the particular form of l_* does not matter for this property, what is important is that $l_*(F)$ is covariantly constant with respect to the connection A, which is the case since we assume F to be covariantly constant.

The choice of an appropriate star-function l_* is however crucial to ensure the invariance of the above action under HS Weyl transformations, i.e. gauge transformations generated by w. To see that, let us first point out that the action (3.14) can be interpreted as a trace. Indeed, interpreting A as a Fedosov connection the space $\mathcal{S}(\mathcal{X})$ of functions on $T^*\mathcal{X}$ can be endowed with a star-product, via

$$f\star g:=(F*G)|_{y=0}\,, \tag{3.16}$$

where $F=F(f)$ and $G=G(g)$ are the unique covariantly constant sections such that $F|_{y=0}=f$ and $G|_{y=0}=g$ (see appendix E). Then for any $f\in\mathcal{S}(\mathcal{X})$ of compact support, the operation

$$\mathrm{tr}_A(f):=\int_{\mathcal{X}}\int_{p-\text{fiber}}\mu(F;A,\dots,A)\,, \tag{3.17}$$

defines a trace, in the sense that it verifies

$$\mathrm{tr}_A(f\star g)=\mathrm{tr}_A(g\star f)\,, \tag{3.18}$$

for any other $g\in\mathcal{S}(\mathcal{X})$. This cyclicity property also follows directly from the flatness of A, the covariant constancy of F and G, and the cocycle condition obeyed by μ, upon discarding

[19] Recall that the Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$ is embedded in the Weyl algebra $\mathcal{A}_{2n}$ as the subspace of quadratic elements.

total derivative terms. The action (3.14) can then be re-written as

$$S[A,F] = \mathrm{tr}_A\big(l_\star(f)\big)\,, \tag{3.19}$$

with $f = F|_{y=0}$, and its variation under HS Weyl transformations reads

$$\delta_w S = 2\,\mathrm{tr}_A\big(l'_\star(f)\star f\star w_0\big) = \int_{\mathcal{X}}\int_{p-\text{fiber}} 2\,\mu\big(l'_*(F)*F*w;A,\ldots,A\big)\,, \tag{3.20}$$

where $w_0 = w|_{y=0}$. As in the Segal case, one can see that the choice $l_*(F) = \Theta_*(F)$—the Heaviside function, guarantees the invariance of the action under HS Weyl transformations, since $l'_*(x)*x = \delta_*(x)*x = 0$ (at least formally, see appendix B for more details). On top of that, this choice also implies that our action reduces to Segal's around flat space. Indeed, for $\mathcal{X} = \mathbb{R}^n$, the choice $A^{(0)} = dx^\mu\delta^a_\mu p_a$ can be made globally, so that the action reduces to

$$S[f] = S[A = A^{(0)}, F = F(f)] = \int_{\mathbb{R}^{2n}} \Theta_\star(f)\,, \tag{3.21}$$

upon using the normalization property (*ii*) of μ, and the fact that the star-product $\star$ simply becomes the Moyal–Weyl star-product in x and p for this particular choice of connection $A = A^{(0)}$.

The above analysis implies that the system is gauge invariant under the transformations generated by the gauge parameters ξ and w. Let us dwell on the interpretation of the system: action (3.14) is understood as a functional defined on the space of solutions of the off-shell system (3.1). As we are going to see in the next subsection, using the gauge freedom one can set A to be a fixed connection while solutions for F are 1:1 with the unconstrained configurations $f(x,p) = F|_{y=0}$. Consequently, the functional (3.14) gives a globally well-defined action on the configuration space of unconstrained $f(x,p)$.

4. Gauge conditions, field redefinitions, and background fields

The action (3.14) supplemented with the off-shell constraints (3.1) and gauge transformations (3.2) and (3.3) of the parent Segal system defines the action of CHS gravity in a covariant and coordinate-independent way. However, in this formulation the system involves an overcomplete set of fields, which effectively reduces to the minimal one only upon taking into account off-shell constraints and algebraic gauge transformations. Below we discuss this procedure in more details, and identify several useful gauge conditions.

4.1. Segal gauge

Our first task is to demonstrate that locally any solution to the zero-curvature equation (3.1) is equivalent to one where $A = A^{(0)}$ with some fixed $A^{(0)}$. Recall that we consider connections A whose piece linear in p is invertible, i.e. e^a_μ entering $dx^\mu e^a_\mu(x)p_a$ is invertible. In other words, we want to prove that locally all flat connection on the Weyl bundle with an invertible e^a_μ belong to the same gauge orbit. The parent version of HS diffeomorphisms act on A and F as

$$\begin{aligned} A' &= e_*^{-\frac{\lambda}{\hbar}} * (\hbar d + A) * e_*^{\frac{\lambda}{\hbar}}\,, & F' &= e_*^{-\frac{\lambda}{\hbar}} * F * e_*^{\frac{\lambda}{\hbar}}\,, \\ \delta_\lambda A &= d\lambda + \frac{1}{\hbar}[A,\lambda]_*\,, & \delta_\lambda F &= \frac{1}{\hbar}[F,\lambda]_*\,, \end{aligned} \tag{4.1}$$

where in the second line we list the infinitesimal version. In the present local analysis, we take $A^{(0)} = dx^\mu\delta^a_\mu p_a$. It is convenient to denote space-time coordinates by x^a so that $A^{(0)} = dx^a p_a$.

Using the above gauge transformations with λ of the form $\lambda_a(x)y^a$, one can set to zero the y,p-independent term in A. Then $\lambda = \lambda_a^b y^a p_b$ allows us to set $e_b^a = \delta_b^a$. Furthermore, the flatness condition for A sets to zero the antisymmetric part of h_{ab} entering A as $dx^a h_{ab} y^b$ in A. The symmetric part is then set to zero by gauge transformations with λ of the form $\lambda_{ab}y^a y^b$ so that one can assume that $A = dx^a p_a +$ terms of higher order in y,p.

Suppose that A' is another flat connection with invertible e_b^a. As above we can also assume that it starts with $dx^a p_a$. Introduce the following degree:

$$\deg(y) = 1 = \deg(p)\,, \qquad \deg(\hbar) = 2\,, \tag{4.2}$$

which is precisely the degree used in Fedosov quantization, and hence we will refer to it from now on as the Fedosov degree. Expanding A and A' according to this degree as $A = A_{(1)} + A_{(2)} + \cdots$, and similarly for A', one has $A_{(1)} = A'_{(1)}$.

We then continue by induction. To this end, let us assume that $A_{(l)} = A'_{(l)}$ for all $l \leqslant k$. The zero-curvature equations for A and A' imply

$$\delta(A'_{(k+1)} - A_{(k+1)}) = 0\,, \qquad \delta \equiv -\frac{1}{\hbar}[A_{(1)}, \cdot]_* = dx^a \frac{\partial}{\partial y^a}\,, \tag{4.3}$$

where following Fedosov we have introduced a nilpotent operator δ. Because the cohomology of δ is trivial in nonvanishing form-degree, it follows $A'_{(k+1)} - A_{(k+1)} = \delta\lambda_{(k+2)}$ for some $\lambda_{(k+2)}$. Applying gauge transformation (4.1) with $\lambda = \lambda_{(k+2)}$ to A' one finds $A - A'$ is of degree $k+2$ or higher. In particular, taking $A = dx^a p_a$, one finds that this gauge is locally reachable.

Despite this gauge being reachable only locally, it is very instructive. In particular, it is clear that the covariant constancy condition $dF + \frac{1}{\hbar}[A^{(0)}, F]_* = 0$ has a unique solution $F = f(x+y,p)$ satisfying $F|_{y=0} = f$ for any unconstrained $f = f(x,p)$.

4.2. *Metric-like gauges*

Segal's gauge just discussed is the simplest example of a gauge where all the independent fields are contained in F, while A is set to a background value. We refer to such gauges as metric-like ones.

A class of globally well-defined connections on the Weyl bundle can be constructed starting with a given torsion-free affine connection. More specifically, let

$$\varpi = e^a \boldsymbol{P}_a + \Gamma^a{}_b \boldsymbol{T}^b{}_a\,, \qquad \boldsymbol{P}_a = p_a\,, \qquad \boldsymbol{T}^a{}_b := y^a p_b\,, \tag{4.4}$$

verify

$$de^a + \Gamma^a{}_b e^b = 0\,, \tag{4.5}$$

so that its curvature is given by

$$d\varpi + \tfrac{1}{2\hbar}[\varpi, \varpi]_* = R^a{}_b \boldsymbol{T}^b{}_a\,. \tag{4.6}$$

It is a standard statement [46] (see also [40, 47] for precisely this setup and appendix E for a proof of a more general statement) that such a connection has a unique completion A such that $A_{(0)} = 0, A_{(1)} + A_{(2)} = \varpi$ and $hA_{(l)} = 0$ for $l > 2$, where we again use the decomposition in Fedosov degree (4.2). Here h is a contracting homotopy for δ, given by

$$h = \tfrac{1}{N} y^a e_a^\mu \imath_{\partial_\mu}\,, \tag{4.7}$$

where $\imath_{\partial_\mu}$ denotes the interior product by the vector field ∂_μ, and N is the operator counting the sum of the form degree as well as the degree in y. Note also that the completion is such that all $A_{(l)}$ are linear in p. The first few orders read as

$$A = e^a \boldsymbol{P}_a + \Gamma^a{}_b \boldsymbol{T}^b{}_a - e^a \Big(\tfrac{1}{3} R_{ab}{}^d{}_c\, y^b y^c p_d + \tfrac{1}{12} \nabla_b R_{ac}{}^e{}_d\, y^b y^c y^d p_e + \left[\tfrac{1}{60} \nabla_b \nabla_c R_{ad}{}^f{}_e + \tfrac{2}{45} R_{ab}{}^g{}_c R_{de}{}^f{}_g \right] y^b y^c y^d y^e p_f + \cdots \Big), \tag{4.8}$$

where the ... denote corrections of higher order in y, but do not contain any terms in $\hbar$ (this is due to the fact that ϖ is linear in p, see appendix E). Similarly, the first few orders of a covariantly constant section F such that $F|_{y=0} = f(x,p)$ are given by

$$F = f + y^a \nabla_a f + \tfrac{1}{2} y^a y^b \left(\nabla_a \nabla_b + \tfrac{1}{3} R_{ab}{}^c{}_d\, p_c \tfrac{\partial}{\partial p_d} \right) f + \cdots, \tag{4.9}$$

where the ... represent higher order corrections in both y and $\hbar$.

It can be useful to take ϖ to be a metric-compatible connection. In this case, A constructed above and F determined by $f = \frac{1}{2} \eta^{ab} p_a p_b$, where η is a Minkowski metric, describe a gravitational background. In particular, given a frame e, the flat connection A is entirely determined by the metric $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$.

The gauge just constructed can be considered as a covariantized and globally well-defined version of the Segal gauge. A reason to call it metric-like is that the independent fields encoded in $f(x,p)$ are totally symmetric tensors (a covariantized version of Fradkin–Tseytlin fields). This gauge is useful in the analysis of the propagation of CHS fields on gravitational backgrounds, and it was already employed in this context in [40].

By employing a fixed metric-like gauge, the covariantized Segal action becomes a functional of the metric-like CHS fields encoded in $f(x,p)$. Because this action is gauge-invariant, the gauge-fixed actions corresponding to gauge-equivalent background connections A and A' should be related by a field redefinition.

Note also that by a suitable gauge transformation one can set $e^a_\mu = \delta^a_\mu$, i.e. pick a coordinate local frame. In this case, the flat connection reads

$$A = dx^\mu \left(p_\mu + \Gamma^\lambda_{\mu\nu} y^\nu p_\lambda - \tfrac{1}{3} R_{\mu\nu}{}^\lambda{}_\sigma\, y^\nu y^\sigma p_\lambda + \cdots \right). \tag{4.10}$$

Let us also mention that, due to the $\mathfrak{sp}(2n,\mathbb{R})$-invariance of cocycle μ, (3.17), the connection Γ never appears alone in the expression of the action. Indeed, it is the component of A along a quadratic element of the Weyl algebra, and hence belongs to its $\mathfrak{sp}(2n,\mathbb{R})$ subalgebra. In other words, Γ will appear in the final action only through the covariant derivative ∇, or its curvature R.

The above construction of a flat connection starting from a curved one is an instance of the slightly more general mechanism of connection flattening in the Weyl algebra, which is encompassed in the following proposition.

Proposition 4.1. *Let $\mathcal{D}$ be a connection on the Weyl bundle $W(\mathcal{X})$, whose connection 1-form is $\mathcal{A}_{2n}$-valued and acts in the adjoint. Let in addition the Fedosov degree 1 piece of $\mathcal{D}$ be an invertible vielbein. Then there exists a 1-form $w \in \Omega^1(\mathcal{X}) \otimes \Gamma(\widehat{W}(\mathcal{X}))$ such that $\mathcal{D} + \frac{1}{\hbar}[w,-]_*$ is a flat connection. Moreover, any $f \in \Gamma\big(S(T\mathcal{X})\big)$ has a unique completion to a section $F \in \Gamma(W(\mathcal{X}))$ such that $\mathcal{D}F + \frac{1}{\hbar}[w,F]_* = 0$ and $F|_{y=0} = f$. We refer to F as to a covariantly-constant lift of f.*

Say that the connection on the Weyl bundle is (locally) given by $\mathcal{D} = d + \frac{1}{\hbar}[\varpi,-]_*$, then the flat connection is given by $d + \frac{1}{\hbar}[A,-]_*$ with $A = \varpi + w$. For instance, in the previous example we considered an affine connection encoded by the $\varpi = e^a \boldsymbol{P}_a + \Gamma^a{}_b \boldsymbol{T}^b{}_a$ valued in $\mathfrak{igl}(n,\mathbb{R})$, but in principle, one could consider more general, nonlinear connections as a starting point. The proof of this statement is a minor modification of the proof that any symplectic connection can be lifted to a flat connection on the Weyl bundle of a symplectic manifold [46, theorem 3.2] (see also [98, theorem 2]) and is therefore relegated to appendix E.

4.3. Frame-like gauges

In some sense, opposite to metric-like gauges are frame-like ones. In these gauges, all the dynamical fields are contained in the A-field and are interpreted as components of a connection 1-form.

To see this, let us again use the Fedosov degree (4.2) and start with a generic solution (A,F) of the parent system (recall that $e^a_\mu(x)$ in the term $dx^\mu e^a_\mu(x)p_a$ in A is assumed invertible). In addition, we now also assume $g^{ab}(x)$ in the term $\frac{1}{2}g^{ab}p_ap_b$ in F to be invertible and to have Lorentzian signature. Performing a finite gauge transformation (4.1) with λ of the form $h_a(x)y^a$, one can achieve $F_{(1)}=f_a(x)y^a$, i.e. remove the term linear in p_a[20]. In so doing, the transformed A gets in addition the nonvanishing $A_{(0)}=A_{(0)}(x)$ contribution. In the next step, we perform a gauge transformation with λ of the form $\lambda^a_b y^b p_a$ in order to set $F_{(2)}$ to be $\frac{1}{2}\eta^{ab}p_ap_b$, where η^{ab} is the inverse of the standard Minkowski metric.

We then proceed by induction. Suppose that by further gauge transformations we succeeded to set the p-dependent parts of $F_{(l)}|_{y=0}$ to zero for all $l\leqslant k$ save for $l=2$. The p-dependent part of $F_{(k+1)}|_{y=0}$ can be then gauged away by parameters of the form $\sum_m \hbar^m y^a\lambda_a^{b_1\ldots b_{k-2m}}p_{b_1}\ldots p_{b_{k-2m}}$ (i.e. of degree $k+1$ and linear in y). By degree reasoning, such gauge transformations cannot affect $F_{(l)}$ with $l\leqslant k+1$. At the same time, we can eliminate the p-independent term in $F_{(k+1)}|_{y=0}$ by the leading (in $\hbar$) term of the gauge transformation. However, in so doing one can get nonvanishing contributions in $F_{(k+1)}$, which are proportional to $\hbar$. The procedure can then be iterated, giving a p-independent $F_{(k+1)}$.

The induction then implies that the gauge where $F|_{y=0}=D(x)+\frac{1}{2}\eta^{ab}p_ap_b$ is reachable. In other words, the configuration of the parent system is entirely determined by the configuration of the connection A (although, strictly speaking D is not captured by A). Indeed, proposition 4.1 implies that $D(x)+\frac{1}{2}\eta^{ab}p_ap_b$ admits a unique covariantly constant lift F satisfying $F|_{y=0}=D(x)+\frac{1}{2}\eta^{ab}p_ap_b$.

A natural question is then which components of A can be taken as independent fields. It turns out that the minimal set is given by the HS frame field encoded in

$$E(p)=dx^\mu E_\mu(x,p)=dx^\mu\left(a_\mu(x)+e^a_\mu p_a+\cdots\right). \tag{4.11}$$

For the covariance we also take a fixed torsion-free Lorentz connection $\Gamma=dx^\mu\omega_\mu{}^a{}_b\,y^bp_a$. Starting with

$$\varpi=dx^\mu E_\mu(x,p)-dx^\mu(\partial_\mu a_\nu)e^\nu_a y^a+\Gamma\,,\qquad e^a_\mu e^\mu_b=\delta^a_b\,, \tag{4.12}$$

a minor modification of the proof of the proposition 4.1 allows one to construct a flat connection A satisfying $A|_{y=0}=E(p)$ (and an extra condition, see appendix E). An alternative proof, is given in appendix F.

We have just seen that the HS frame field (whose configurations are 1:1 with metric-like CHS fields if one takes the totally symmetric components of the HS frame) serve as the initial data for the A-field. In particular, the covariantized action (3.14) can be written in this gauge as a functional of the HS frame field $E(p)$. We conclude that this approach also generate a frame-like description of the Segal action.

[20] Strictly speaking, this is true under the assumption that all $s\neq 2$ fields are small compared to $s=2$ field. Otherwise, the respective series may diverge.

It is worth mentioning that the parent formulation was initially developed to explicitly relate metric-like and frame-like formulations. In particular, the Lagrangian version of the parent formalism [99, 100] allows one to systematically derive a frame-like formulation starting from the metric-like one, so that it is not surprising that these formulations are reproduced through different gauges of the parent system.

To illustrate the relation between the frame-like and the metric-like gauges, let us fix a gravitational background described by the frame e^a and the Lorentz connection ω^{ab}, so that the metric is $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, and consider a linearized spin-s field $\Phi^{a(s)}(x)$ on this background described in the metric-like gauge. We restrict ourselves to a single CHS field in F since the argument is about free fields for simplicity. More precisely, the configuration for A and F reads as

$$A = e^a \boldsymbol{P}_a + \tfrac{1}{2}\,\omega^{ab}\boldsymbol{L}_{ab} + \cdots\,, \qquad F = \tfrac{1}{2}p^2 + \cdots + \Phi^{a(s)}\,p_a \ldots p_a + \cdots \tag{4.13}$$

where ... denotes the terms that complete the initial data of A and F to a solution of the parent system (3.1). Now, if we perform a gauge transformation with

$$\xi = \tfrac{1}{2}\,\Phi^{a(s)}\,y_a p_a \ldots p_a + \cdots\,, \tag{4.14}$$

we get as a result

$$A = e^a \boldsymbol{P}_a + \tfrac{1}{2}\,\omega^{ab}\boldsymbol{L}_{ab} - \tfrac{1}{2}\,e_a\,\Phi^{ab(s-1)}\,p_b \ldots p_b + \tfrac{1}{2}\,\nabla\Phi^{a(s)}\,y_a p_a \ldots p_a + \cdots\,, \tag{4.15}$$

and

$$F = \tfrac{1}{2}p^2 + \cdots\,, \tag{4.16}$$

i.e. we have moved the linearized spin-s field from F to A. In the last two components of A, we see the CHS vielbein $e^{a(s-1)} = e_m \Phi^{ma(s-1)}$ in a particular gauge followed by its first auxiliary field that is expressed in terms of the first derivative of $\Phi^{a(s)}$.

Let us finally mention that in showing the existence of the metric-like and the frame-like gauges, we only made use of HS diffeomorphisms. It follows that the analogous gauges are reachable in the version of the parent system [47, 97, 100] with HS Weyl transformations dropped, which describes nonlinear gauge transformations of the off-shell massless HS fields with the trace constraint relaxed. Let us note that massless HS fields within a similar framework were discussed in [101].

4.4. Conformal geometry gauge

We now get back to metric-like gauges and demonstrate that with a suitable choice of A one can make the underlying conformal geometry manifest.

The idea of this approach is to observe that the conformal algebra $\mathfrak{so}(n,2)$ can be identified as a Lie subalgebra of $\mathcal{A}_{2n}$, and more specifically of its subalgebra of elements linear in p. In the standard basis, the commutation relations read as

$$[\boldsymbol{D},\boldsymbol{P}^a] = +\boldsymbol{P}^a\,, \qquad [\boldsymbol{L}^{ab},\boldsymbol{P}^c] = \boldsymbol{P}^a\eta^{bc} - \boldsymbol{P}^b\eta^{ac}\,, \tag{4.17a}$$

$$[\boldsymbol{D},\boldsymbol{K}^a] = -\boldsymbol{K}^a\,, \qquad [\boldsymbol{L}^{ab},\boldsymbol{K}^c] = \boldsymbol{K}^a\eta^{bc} - \boldsymbol{K}^b\eta^{ac}\,, \tag{4.17b}$$

$$[\boldsymbol{K}^a,\boldsymbol{P}^b] = \eta^{ab}\boldsymbol{D} - \boldsymbol{L}^{ab}\,, \qquad [\boldsymbol{L}^{ab},\boldsymbol{L}^{cd}] = \boldsymbol{L}^{ad}\eta^{bc} + \text{three more}\,. \tag{4.17c}$$

with $\boldsymbol{P}_a, \boldsymbol{L}_{ab}, \boldsymbol{D}$ and $\boldsymbol{K}_a$ the generators of translations, the Lorentz transformations, dilation and special conformal transformations respectively. They can be represented in $\mathcal{A}_{2n}$ as

$$\boldsymbol{P}_a = p_a\,, \qquad \boldsymbol{L}_{ab} = 2p_{[a}\,y_{b]}\,, \qquad \boldsymbol{D} = y^a\,p_a + \Delta\,, \qquad \boldsymbol{K}_a = y_a\,(y\cdot p + \Delta) - \tfrac{1}{2}\,y^2\,p_a\,, \tag{4.18}$$

where $\Delta \in \mathbb{R}$ is any real number for the moment.

A generic connection ϖ valued in the conformal algebra,

$$\varpi = e^a\,\boldsymbol{P}_a + \tfrac{1}{2}\,\omega^{ab}\,\boldsymbol{L}_{ab} + b\,\boldsymbol{D} + f^a\,\boldsymbol{K}_a\,, \tag{4.19}$$

can be put in a simpler form by gauge fixing its component b to zero, imposing that it is torsionless (so that the spin-connection is expressed in terms of the vielbein) and taking f^a to be the Schouten 1-form[21] P^a, whose components read

$$P_\mu{}^a := \tfrac{1}{n-2}\left(R_\mu{}^a - \tfrac{1}{2(n-1)}\,e^a_\mu\,R\right), \tag{4.20}$$

where $R_\mu{}^a = R^{ab}_{\mu\nu}\,e^\nu_b$ and $R = R^a_\mu\,e^\mu_a$, with $R^{ab}_{\mu\nu}$ the Riemann curvature of ω. The curvature of this gauge fixed version of the conformal connection,

$$\varpi = e^a\,\boldsymbol{P}_a + \tfrac{1}{2}\,\omega^{ab}\,\boldsymbol{L}_{ab} + P^a\,\boldsymbol{K}_a\,, \tag{4.21}$$

takes the simple form

$$d\varpi + \tfrac{1}{2\hbar}\,[\varpi,\varpi]_* = \tfrac{1}{2}\,C^{ab}\,\boldsymbol{L}_{ab} + (\nabla P^a)\,\boldsymbol{K}_a\,, \tag{4.22}$$

where $C^{ab} := R^{ab} - 2\,e^{[a}P^{b]}$ is the Weyl 2-form.

We can now apply the flattening procedure of proposition 4.1 to construct a flat connection A starting from the previously described conformal connection ϖ. The first few orders of A are given by

$$\begin{aligned} A = e^a\,\boldsymbol{P}_a + \tfrac{1}{2}\,\omega^{ab}\,\boldsymbol{L}_{ab} + P^a\,\boldsymbol{K}_a + \tfrac{1}{3}\,e^a\,C_{abc}{}^d\,y^b y^c\,p_d \\ + e^a\left(\tfrac{1}{2}\,\nabla_{[b}S_{a]|cd}{}^e + \tfrac{1}{12}\,\nabla_b C_{acd}{}^e\right) y^b y^c y^d\,p_e + \cdots\,, \end{aligned} \tag{4.23}$$

where we introduced the tensor

$$S_{a|bc}{}^d := P_{ae}\,(\delta^e_{(b}\delta^d_{c)} - \tfrac{1}{2}\,\eta^{ed}\eta_{bc})\,, \tag{4.24}$$

and by construction, the higher orders terms in A will be contraction of the covariant derivative of the Weyl tensor C and the Schouten tensor P. Having determined A, we can now turn our attention to the 0-form F, which can be constructed as a covariantly constant lift of an unconstrained $f(x,p)$. The first few orders of F are given by

$$F = f + y^a\,\nabla_a f + \tfrac{1}{2}\,y^a y^b\,\Big(\nabla_a\nabla_b + [\tfrac{1}{3}\,C_{adb}{}^c + 2\,S_{a|bd}{}^c]\,p_c\,\tfrac{\partial}{\partial p_d}\Big)\,f + \cdots\,, \tag{4.25}$$

where the dots indicate corrections of higher order in y and $\hbar$[22].

Now let us focus on the spin-2 case. That is, we consider that the 0-form F is simply the completion of $\frac{1}{2}\,p^2 = \frac{1}{2}\,\eta^{ab}\,p_a p_b$,

$$F = \tfrac{1}{2}\,p^2 + y^a y^b\,(\tfrac{1}{6}\,C_a{}^c{}_b{}^d + S_{a|b}{}^{cd})\,p_c p_d + \cdots\,, \tag{4.26}$$

into a covariantly constant section. With A being the completion of the normal Cartan connection into a flat connection of the Weyl bundle, this gauge is a frame-like one. The gauge transformations generated by a parameter w which is the lift of the Weyl parameter σ,

$$w = \sigma + y^a\,\nabla_a\sigma + \tfrac{1}{2}\,y^a y^b\,\nabla_a\nabla_b\sigma + \tfrac{1}{6}\,y^a y^b y^c\,\big(\nabla_a\nabla_b\nabla_c + 2\,S_{a|bc}{}^d\,\nabla_d\big)\sigma + \cdots\,, \tag{4.27}$$

[21] This last condition follows from imposing that the curvature of ϖ along the Lorentz generators $F[\varpi]^{ab}$ is traceless in the sense that $F[\varpi]^{ab}_{\mu\nu}\,e^\nu_b = 0$. This connection is called the normal Cartan connection [102, definition 1.6.7 or 3.1.12], see also [103, section 2] and [104, section 3.2] for more details.

[22] Note that $\hbar$ corrections will appear in F only if f contains terms which are at least quadratic in p.

reads

$$\delta_w F = p^2\,\sigma + p^2\,y^a\,\nabla_a\sigma + \mathcal{O}(y^2)\,, \tag{4.28}$$

thereby signaling F does encode a conformal spin-2 field. Since the connection A also contains a conformal connection, and therefore also describes a conformal spin-2 field, let us check that their gauge transformations are compatible. By this, we mean to check whether it is possible to find a gauge parameter ξ such that the frame-like gauge is preserved, i.e.

$$\delta_{\xi,w} F = \mathcal{O}(y^2)\,, \tag{4.29}$$

so that the connection A is transformed, and only the completion of $\frac{1}{2}p^2$ in F, i.e. terms of order 2 or higher in y and $\hbar$, are affected. Inspecting the above equation at the first few orders, one finds that ξ should take the form

$$\xi = \sigma\,\boldsymbol{D} + \partial_a\sigma\,\boldsymbol{K}^a\,, \tag{4.30}$$

and hence it implements the usual gauge transformation of the normal conformal connection[23].

4.5. Higher spin gauge

It is sometimes convenient to rearrange a theory in terms of the symmetries of one of its (maximally symmetric) backgrounds. In the CHS gravity case, this corresponds to $F^{(0)} = \frac{1}{2}p^2 \equiv \frac{1}{2}\eta^{ab}\,p_a p_b$. As it was already discussed in section 2.3, the global symmetry algebra of this background is exactly the HS algebra $\mathfrak{hs}(\Box\phi)$ of higher symmetries of Laplacian [24, 85] (more or less by definition). The latter also fixes the conformal weight Δ in (4.18) accordingly.

It turns out that one can reconstruct field configurations in the frame-like gauge of section 4.3 in terms of a connection of the HS algebra. To this end, let us fix an embedding of HS algebra as a subspace in $\mathcal{A}_{2n}$, together with a projection to this subspace and take as ϖ a connection with values in the subspace. It can be then lifted to a flat connection A by applying proposition 4.1. What is important is that independent fields sit in the HS frame part (which is the y-independent part of ϖ) but the completion does not affect this part and hence gives a particular lift of the HS frame to a flat A. The F field is then reconstructed by a covariantly constant lift of $\frac{1}{2}p^2 + D(x)$.

In this way we conclude that we succeeded to parameterize solutions to the parent system in terms of a connection of HS algebra. Of course this parametrization is much more redundant than the one in terms of HS frame. Moreover, it is also not clear how to explicitly identify the HS algebra as a gauge algebra in this setup. It would also be very interesting to come up with an appropriate HS extension of the normal Cartan connection.

4.6. Gauge symmetries vs. field redefinitions

Given that the action of CHS gravity $S[A,F]$ depends on two fields that are subject to constraints (3.1), it feels necessary to dwell on possible interpretations of such an action. As we have already discussed, one can make use of a metric-like gauge where the action becomes a functional of the initial data $f(x,p)$ only. Of course, it still depends on a fixed connection A but it is considered as a parameter, or better, a background field. Thanks to the gauge invariance of the covariant action, the change of A leads to a field redefinition in terms of $f(x,p)$. This should

[23] The fact that the coefficient of $\boldsymbol{K}_a$ in the gauge parameter is the derivative of the Weyl parameter σ is a consequence of the fact that we have gauge fixed b (the gauge field associated with dilation) to zero earlier.

be compared with the standard background field method (for gravity), see e.g. [33]. Note that a version of the background field method in precisely this context was used in [40].

If, on the contrary, we employ a frame-like gauge, where the parent field configuration is determined solely by a HS frame $E(p)$, the action becomes a functional of E which is unconstrained. Alternatively, one can go for a more redundant description in which A is parametrized by a connection of the HS algebra.

For applications it may be useful to distinguish between three different types of gauge symmetries of the parent Segal system:

(1) Those generated by parameters ξ that are *not* covariantly constant ($d\xi + \frac{1}{\hbar}[A,\xi]_* \neq 0$) correspond to field redefinitions: they allow one to move components of A and F into one another (as illustrated at the end of section 4.3). In fact, one has to consider the quotient of all ξ by the covariantly constant ones;
(2) Those generated by *covariantly constant* parameters ξ and w are a covariant version of Segal's original gauge symmetries, in the sense that they affect only the 0-form F via the commutator and anti-commutator respectively;
(3) Those generated by covariantly constant gauge parameters ξ and w and that also preserve a given vacuum $F^{(0)}$ correspond to global symmetries and define a HS algebra, $\mathfrak{hs}(F^{(0)})$.

5. Conclusions and discussion

We constructed a covariant action for the simplest class of conformal HS gravities, which can be associated with the free scalar conformal matter, $\square\phi = 0$. CHS gravity is the theory of the background conformal fields that couple to 'single-trace' operators, or to put it simply, bilinear operators $J_s = \phi\partial\ldots\partial\phi + \ldots$, most of which are conserved (HS) tensors.

The constructions of CHS gravity proposed by Tseytlin and Segal are closely related and prove the theories to be well-defined (at least in terms of a perturbative expansion around flat space background). It goes without saying that theories of gravity should admit manifestly covariant, coordinate- and background-independent formulations. Addressing this question is the goal of the present paper. It is quite amusing that the action of CHS gravity requires such advanced constructions from deformation quantization as Shoikhet–Tsygan–Kontsevich formality that gives a proper measure for the invariant trace on the algebra of quantum observables, using the Feigin–Felder–Shoikhet cocycle. Somewhat related links to the same formality have already been observed [105], in particular, for Chiral HS gravity [106–108].

The induced action for CHS gravity can be derived from a simple particle model, as mentioned in Segal's paper [24] (see also [54]), or discussed in more details in [55]. As it turns out, the action (3.14) can also be obtained from a particle model. Indeed, the main ingredient used to construct this action, namely the Feigin–Felder–Shoikhet cocycle, admits a representation as a correlation function in a particular one-dimensional sigma-model, often called topological quantum mechanics [53, 109]. More specifically, this model is the simplest example of AKSZ type [110], namely, the $1d$ AKSZ model whose target space is the BFV–BRST extended phase space of a constrained Hamiltonian systems[24]. The underlying BFV–BRST system is precisely the BFV–BRST reformulation of the particle model [24] underlying CHS theory and reviewed in appendix A. The Fedosov extension of the particle model again leads to an

[24] Such AKSZ-like models were introduced in [111] and shown to produce the BV formulation for the respective extended Hamiltonian action. If the Hamiltonian is non-trivial, such models have a slightly more general structure but in the case at hand the Hamiltonian is trivial and the model is of AKSZ type.

extended AKSZ sigma model which produces the invariant trace (and hence the FFS cocycle itself) as a correlation function [53, 112]. Note that the Fedosov-like extension itself can be understood as a passage to the extended BFV–BRST system [113]. Moreover, it is precisely the BFV–BRST system underlying the parent reformulation [47, 48] we employ in this work.

Extensions and generalizations of the present work should exist along several lines: (a) one can choose different vacua $F^{(0)}$ for F, which is equivalent to having the same type of matter, e.g. scalar, but with different conformally-invariant equations, e.g. $\Box^k\phi = 0$, $k > 1$ for the scalar matter corresponds to $F^{(0)} = (p^2)^k$; (b) one can choose different types of matter, e.g. fermion ψ, or, more generally, a mixed-symmetry (spin-)tensor field, see [114] for the discussion of the CHS gravity based on the free fermion (called Type-B) and [115, 116] for further extensions; (c) supersymmetric extensions should also be possible and be based on the Clifford–Weyl algebra, see the recent [45, 61] for $\mathcal{N} = 1$.

We expect that the approach of this paper provides an efficient way to attack some of the problems of CHS fields that have been around for a while: whether conformal gravity is a consistent truncation of CHS gravity [39–41]?; what are the gravitational backgrounds that admit free CHS fields [39–45]?; the structure of (HS) Weyl anomaly and, hence, the problem of quantum consistency of CHS gravity.

It is well-known that the deformation quantization of a symplectic manifold M up to a natural equivalence is in one-to-one with characteristic classes $\Omega[\hbar] = \Omega_0 + \hbar\Omega_1 + \cdots$, where $\Omega_i \in H^2(M,\mathbb{C})$ and Ω_0 is the class of the symplectic form. In the case of a cotangent bundle Ω_0 is trivial. There is a simple deformation of the off-shell parent system (3.1)–(3.3), which allows one to put CHS fields on an external electromagnetic background, e.g. Dirac string. One needs a nontrivial class $H^2(\mathcal{X},\mathbb{C})$ of the base manifold $\mathcal{X}$ itself, which can be added to the rhs.

$$dA + \tfrac{1}{2\hbar}[A,A]_* = \Omega[\hbar]\,. \tag{5.1}$$

This is one simple generalization of CHS gravity that is not accessible in a local chart, where one can always impose Darboux coordinates. The possibility to add de Rham cohomology classes as deformations to HS systems seems to be a quite generic feature [117, 118].

A closely related idea is that a general solution $f = c_1\Theta(x) + c_2$ to $f'(x)x = 0$ contains the constant term. Though it does not make any contribution in Darboux coordinates, for a general compact symplectic manifold the quantity $\mathrm{tr}_A(1)$ leads to a particular case of Fedosov/Nest-Tsygan index theorem [119–121]. However, the cotangent bundle is non-compact. It would be interesting to see if some of the index theorems admit HS extensions. For example, Euler characteristic is the second conformal invariant in 4d and appears on equal footing with the Weyl gravity action in the studies of conformal anomalies. It remains an open question whether it admits a HS extension and what is the interpretation of the corresponding topological invariant.

As it was already pointed out in [24] and explored in [75, 76], a CHS gravity can be coupled to the matter it originated from (via the effective action approach or via the Segal approach). In the Segal approach the corresponding coupling is simply

$$S[H,\phi] = S_{\mathrm{CHS}}[H] + \langle\phi|\widehat{H}\phi\rangle\,.$$

It would be interesting to find its covariant extension along the lines of the present paper.

Note that any CHS gravity can be truncated to its low spin (not higher than spin-two) subsector by setting all HS fields to zero. Nevertheless, a HS extension nicely fits the deformation quantization framework: it is natural to consider all differential operators, which is an associative algebra (Weyl algebra), rather than to restrict to vector fields (see also [82, 122] wherein similar ideas are advocated). The low spin subsector of CHS gravity allows us to make a bridge to conformal (super-)gravities.

Concerning the overlap between CHS gravities and conformal (super)gravities, let us mention a tremendous work done in [123, 124] that culminated in the complete action of the maximal $\mathcal{N}=4$ conformal gauged supergravity. Curiously, this action contains an arbitrary function of scalars, see [125] for the recent discussion. It is not clear how to generate this function via the effective action idea since there does not seem to be possible to introduce this ambiguity into $\mathcal{N}=4$ SYM coupled to background conformal supergravity fields [126]. It would be very interesting to see if the approach advocated in this paper, i.e. HS geometry as deformation quantization, can explain this ambiguity and extend it to HSs.

Another interesting closely related class of theories are self-dual truncations of CHS gravity, which admit a natural twistor space formulation [127, 128]. These theories are specific to four dimensions. Since these theories are much simpler than the full CHS gravity, it can be instructive to see how their spacetime actions, which, in principle, are derivable from twistor space, can be formulated within our approach. Similarly, one can try to construct a covariant action for Chiral HS gravity [9–12], which at present is available either in the light-cone gauge or for certain subsectors [14, 15] only.

Another interesting issue is whether the covariant action (and hence the underlying FFS cocycle) can be systematically derived within a purely field-theoretical framework. Indeed, starting from the Segal action one should be able to reconstruct its parent reformulation, and hence reconstruct an appropriate version of FFS cocycle, using the approach of [99, 100]. A slightly alternative field-theoretical interpretation of the covariant action and its underlying cocycle is in terms of a suitable BRST-invariant presymplectic structure (see [104, 118, 129, 130] for more details on the presymplectic BV-AKSZ approach). This would signal an intriguing relation between the geometry of local gauge theories and algebraic structures underlying the deformation quantization.

Let us also point out that the way CHS gravity emerges in the Segal construction is somewhat similar to the IKKT model based on a HS algebra studied e.g. in [16, 20, 22]. Both the Segal construction and the HS-IKKT model [20–22] are examples of non-commutative field theories. Lorentz invariance is violated in generic non-commutative theories due to explicit dependence of Poisson structure $\theta^{\mu\nu}(x)$ on x. In the Segal construction, it is the phase-space that is quantized and there is no explicit violation of Lorentz symmetry ($\theta^{\mu\nu}(x)$ pairs up x–p and vanishes for x–x). In addition, the spacetime Lagrangian is obtained via tracing out or averaging over the p-fiber, which does not violate Lorentz symmetry (in fact, the averaging over p, as we showed, can be performed in a general covariant manner). In the HS-IKKT model, the trick is in having a nontrivial fibration over the spacetime that is again averaged over without having to violate Lorentz symmetry.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. BRST form of Segal

Consider the BFV phase space with coordinates (x^a, p_b) of ghost number 0, and a ghost pair (c, b), i.e. of respective ghost number $+1$ and -1. We use the language of symbols and star product (for the moment we assume Moyal–Weyl star product). The nonvanishing star-commutators between these coordinates are

$$[x^a, p_b]_* = \hbar\delta^a_b\,, \qquad [c, b]_* = \hbar\,, \tag{A.1}$$

and the algebra of functions in these coordinates is equipped with an anti-involution † defined by

$$\hbar^\dagger = -\hbar, \quad x^\dagger = x\,, \quad p^\dagger = p\,, \quad c^\dagger = c\,, \quad b^\dagger = b\,, \tag{A.2}$$

which verifies $(AB)^\dagger = (-)^{|A||B|}B^\dagger A^\dagger$ where $|\cdot|$ denotes the ghost number. A generic nilpotent Hermitian BFV charge has the form

$$\Omega = c\,F(x,p)\,, \quad \Omega^\dagger = \Omega\,, \tag{A.3}$$

with $F^\dagger = F$ also Hermitian. Now we view Ω as a generating function of fields, and consider the following gauge theory:

$$[\Omega,\Omega]_* = 0\,, \quad \delta_\Xi\Omega = \frac{1}{\hbar}[\Omega,\Xi]_*\,, \qquad \mathrm{gh}(\Xi) = 0\,, \quad \Xi^\dagger = \Xi\,, \tag{A.4}$$

where Ξ is a generating function of gauge parameters. In our case $\Xi = \xi(x,p) + cb\,w(x,p)$ with $\xi^\dagger = \xi$ and $w^\dagger = -w$. In terms of components the gauge transformations read as:

$$\delta_{\xi,w}F = \frac{1}{\hbar}[F,\xi]_* + \{F,w\}_*\,. \tag{A.5}$$

In particular F remains Hermitian. It is easy to see that the above precisely encode the Segal's gauge transformations. It follows that Segal's gauge transformations are precisely the natural symmetries of the constrained Hamiltonian systems describing the particle model. In particular, HS diffeomorphisms correspond to a quantized version of the canonical transformations of the constrained surface, while HS Weyl transformations correspond to redefinitions of the constraint.

The above BFV–BRST interpretation of the off-shell Segal system was proposed in [47] (see also [71, 131]). It is a useful starting point to construct the respective parent reformulation [47], from which the parent Segal system is obtained by gauge-fixing the gauge fields associated to Weyl transformations.

Appendix B. More on the star-Heaviside function

In this appendix, we recall how to evaluate the Heaviside star-function, as introduced and explained by Segal [24]. Let us start with the definition of a star-function: given a usual function which admits an integral representation of the form

$$f(x) = \oint_C dt \widetilde{f}(t)\, e^{tx}\,, \tag{B.1}$$

where C is some contour in the complex plane, and $\widetilde{f}(t)$ a given function, the corresponding star-function will be defined as

$$f_*(a) = \oint_C dt \widetilde{f}(t)\, e_*^{ta}\,, \tag{B.2}$$

for any elements $a \in \mathcal{A}_{2n}$ of the Weyl algebra and where

$$e_*^{ta} = \sum_{k=0}^{\infty} \tfrac{t^k}{k!}\, a^{*k}\,, \qquad t \in \mathbb{C}\,, \qquad a^{*k} := \underbrace{a * \cdots * a}_{k\,\text{times}}\,, \tag{B.3}$$

is the star-exponential. Any such star function can be expanded as a formal power series in $\hbar$, of the form [24, section 5.2]

$$f_*(a) = \sum_{n=0}^{\infty} \hbar^{2n} \sum_{k=2}^{2n} f^{(k)}(a)\, p_{n,k}(a) \tag{B.4}$$

where $f^{(k)}$ denotes the kth derivative of f and $p_{n,k}(a)$ are monomials of k in the first $4n$ derivatives of a with respect to the variables of the Weyl algebra. In particular, the Heaviside star-function is given in terms of the integral representation

$$\Theta_*(a) := \lim_{\epsilon \to 0^+} \tfrac{1}{2i\pi} \int_{-\infty}^{+\infty} \tfrac{d\tau}{\tau - i\epsilon}\, e_*^{i\tau a}\,, \tag{B.5}$$

and admits a similar expansion in $\hbar$.

The crucial property of the Heaviside star-function is that, according to the expansion (B.4), it verifies

$$\Theta'_*(a) * a = \delta_*(a) * a = 0\,, \tag{B.6}$$

since the derivative of the Heaviside distribution is the Dirac distribution. Note that the above identity should be understood in the sense of distributions. Fortunately, this is enough to prove the invariance under HS Weyl transformations of the Segal action (2.32), since its variation reads

$$\delta_w S[F] = 2 \int d^n x\, d^n p\; \delta_*(F) * F * w = 2 \int d^n x\, d^n p\, \big(\delta_*(F) * F\big)\, w = 0\,, \tag{B.7}$$

where the second equality is obtained upon disregarding a total derivative.

Since (B.6) can raise some doubts, let us illustrate that the quantum identity $\delta_*(a) * a = 0$ reduces to the classical one $\delta(a)a = 0$ and its derivatives. Indeed, let us start with the general expansion of $f_*(H)$[25]:

$$f_*(H) = f(H) + \hbar^2 \left(\tfrac{1}{6} H_{AB} H^A H^B f'''(H) + \tfrac{1}{4} H_{AB} H^{AB} f''(H)\right) + \mathcal{O}(\hbar^4)\,. \tag{B.8}$$

Now, we can write down the expansion of $f_*(H) * H$ to the same order:

$$\begin{aligned} f_*(H) * H = {} & f(H) H + \hbar^2 \left(\tfrac{1}{6} H_{AB} H^A H^B f'''(H) + \tfrac{1}{4} H_{AB} H^{AB} f''(H)\right) H \\ & + \hbar^2 \left(\tfrac{1}{2} H_{AB} H^A H^B f''(H) + \tfrac{1}{2} H_{AB} H^{AB} f'(H)\right) + \mathcal{O}(\hbar^4)\,. \end{aligned} \tag{B.9}$$

[25] In this appendix $\{Y^A\}$, with $A = 1, \ldots, 2n$, collectively denotes the $2n$ variables of the Weyl algebra $\mathcal{A}_{2n}$, $\partial_A \equiv \partial/\partial Y^A$ and $H_{AB} \equiv \partial_A \partial_B H$.

Of course, this is just an expansion of $g_*(H)$, where $g(H) = f(H)H$. In particular, we can rearrange it with the help of $g' = f'H + f$, $g'' = f''H + 2f'$, etc to find

$$f_*(H) * H = g(H) + \hbar^2 \left(\tfrac{1}{6} H_{AB} H^A H^B g'''(H) + \tfrac{1}{4} H_{AB} H^{AB} g''(H)\right) + \mathcal{O}(\hbar^4). \tag{B.10}$$

Lastly, we take $f(H) = \delta(H)$ and observe that the leading term is just the classical identity $\delta(H)H = 0$, while the subleading ones are derivatives of it. Obviously, all of this is a consequence of the fact that the map $\rho : f(H) \mapsto f_*(H)$ is a homomorphism from the subalgebra of $\mathcal{C}^\infty(T^*\mathcal{X})$ generated by H to the $*$-product subalgebra of $W(\mathcal{X})$ generated by H. Therefore, ρ maps $f(H)g(H)$ to $f_*(H) * g_*(H)$, which we apply to $f = \delta(H)$ and $g = H$.

Appendix C. Modification of FFS cocycle

Let us denote by Φ the Chevalley–Eilenberg cocycle associated with the FFS cocycle, whose expression is detailed below. For the sake of conciseness, we will rewrite it as

$$\begin{aligned}&\Phi(a_0; a_1, \dots, a_{2n})\\ &\quad = \int_{u\in\Delta_{2n}} \Big[\mathcal{D}(\partial_{y_0}, \partial_{p_0}, \partial_{y_i}, \partial_{p_i}, u)\, a_0(y_0, p_0)\, a_1(y_1, p_1) \dots a_{2n}(y_{2n}, p_{2n})\Big]\Big|_{y_0 = y_i = p_0 = p_i = 0},\end{aligned} \tag{C.1}$$

where $a_0, a_1, \dots, a_{2n} \in \mathcal{A}_{2n}$ are elements of the Weyl algebra, Δ_{2n} is the standard $2n$-simplex which can be defined as

$$\Delta_{2n} = \left\{(u_1, \dots, u_{2n}) \in [0,1]^{2n} \,|\, 0 \leqslant u_1 \leqslant u_2 \leqslant \dots \leqslant u_{2n} \leqslant 1\right\}, \tag{C.2}$$

and $\mathcal{D}(\partial_{y_0}, \partial_{p_0}, \partial_{y_i}, \partial_{p_i}, u)$ is a function of the partial derivatives with respect to the Weyl algebra variables $\{y_0^a, p_{a0}, y_i^a, p_{ai}\}$, with $i = 0, \dots, 2n$ and $a = 1, \dots, n$, and the simplex coordinates u. Explicitly, it is given by

$$\mathcal{D}(\partial_{y_0}, \partial_{p_0}, \partial_{y_i}, \partial_{p_i}, u) = \exp\left[\hbar \sum_{0 \leqslant k < l \leqslant 2n} \left(\tfrac{1}{2} + u_k - u_l\right)\left(\partial_{y_k} \cdot \partial_{p_l} - \partial_{p_k} \cdot \partial_{y_l}\right)\right] \det(\partial_{y_i}, \partial_{p_i}), \tag{C.3}$$

where $\partial_{y_i} \cdot \partial_{p_j} = \frac{\partial}{\partial y_i^a} \frac{\partial}{\partial p_{aj}}$, and $u_0 = 0$ by convention. Note that the determinant part of this operator *does not* acts on the zeroth argument (a_0) of Φ.

From the above cocycle, we can define a new one, simply by not setting the p_{ai} variables to zero in the above expression but to the same value p'_a for all $i = 0, \dots, 2n$, i.e.

$$\begin{aligned}&\widetilde{\Phi}(a_0; a_1, \dots, a_{2n})(p')\\ &\quad = \int_{u\in\Delta_{2n}} \Big[\mathcal{D}(\partial_{y_0}, \partial_{p_0}, \partial_{y_i}, \partial_{p_i}, u)\, a_0(y_0, p_0)\, a_1(y_1, p_1) \dots a_{2n}(y_{2n}, p_{2n})\Big]\Big|_{y_0 = y_i = 0, p_0 = p_i = p'},\end{aligned} \tag{C.4}$$

for any $a_0, a_1, \dots, a_{2n} \in \mathcal{A}_{2n}$.

Lemma C.1. *The map $\widetilde{\Phi}$ defined above is a Chevalley–Eilenberg cocycle of degree $2n$ for the Lie algebra $(\mathcal{A}_{2n}, [-,-]_*)$ associated with the Weyl algebra, with values in its dual whose coefficients are extended to the algebra of polynomials $\mathbb{C}[p'_a]$ in n variables.*

Proof. This simply follows from the fact that $\widetilde{\Phi}$ is obtained by pre-composing each one of the arguments of Φ by an automorphism of the Weyl algebra $\mathcal{A}_{2n}$. Indeed, $\widetilde{\Phi}$ is simply obtained from Φ by shifting its argument by a parameter p', i.e. $a_i(y_i,p_i) \to a_i(y_i,p_i+p')$. In other words,

$$\widetilde{\Phi}(a_0;a_1,\dots,a_{2n})(p') = \Phi\big(T_{p'}(a_0);T_{p'}(a_1),\dots,T_{p'}(a_{2n})\big)\,, \tag{C.5}$$

where we introduced the operator $T_{p'}(a)(y,p) := a(y,p+p')$, for any $a \in \mathcal{A}_{2n}$. This operator is an automorphism of the Weyl algebra: the Moyal–Weyl star-product is invariant under $Sp_{2n} \ltimes \mathbb{R}^{2n}$, the semi-direct product of the symplectic group with the abelian group of translation in $2n$-dimensions, and $T_{p'}$ is nothing but the operator representing the abelian subgroup of n-dimensional translations. Since $\widetilde{\Phi}$ is simply the composition of the cocycle Φ with automorphisms of the Weyl algebra, it follows directly that $\widetilde{\Phi}$ verifies the same cocycle condition as Φ, hence the lemma. □

Next, one can define a multilinear map

$$\mu : \mathcal{A}_{2n} \otimes \mathcal{A}_{2n}^{\wedge n} \to \mathbb{C}[p_a]\,, \tag{C.6}$$

by the formula

$$\mu(a_0|a_1,\dots,a_n)(p) := \tfrac{1}{n!}\,\epsilon_{b_1\dots b_n}\,\widetilde{\Phi}(a_0;a_1,\dots,a_d,y^{b_1},\dots,y^{b_n})(p)\,, \tag{C.7}$$

for any $a_0,a_1,\dots,a_n \in \mathcal{A}_{2n}$. Note that μ verifies[26]

$$\widetilde{\Phi}(f;p_{a_1},\dots,p_{a_n}) = \tfrac{1}{n!}\,\epsilon_{a_1\dots a_n} f|_{y=0}\,, \tag{C.8}$$

for any $f \in \mathcal{A}_{2n}$, as a consequence of the normalisation condition [50, section 4.2, IV] of Φ.

Lemma C.2. *The map μ defined above verifies*

$$\begin{aligned}\tfrac{\partial}{\partial p_a}\varphi_a(a_{-1}|a_0,\dots,a_n) &= \sum_{i=0}^{n}(-1)^i\,\mu([a_{-1},a_i]_*|a_0,\dots,\widehat{a}_i,\dots,a_n)\\ &\quad+\sum_{i<j}(-1)^{i+j}\,\mu(a_{-1};[a_i,a_j]_*,a_0,\dots,\widehat{a}_i,\dots,\widehat{a}_j,\dots,a_n)\,,\end{aligned} \tag{C.9}$$

with

$$\varphi_a(a_{-1}|a_0,\dots,a_n) := \tfrac{(-1)^{n-1}}{(n-1)!}\,\epsilon_{ab_1\dots b_{n-1}}\,\widetilde{\Phi}(a_{-1};a_0,\dots,a_n,y^{b_1},\dots,y^{b_{n-1}})\,, \tag{C.10}$$

and for any $a_{-1},a_0,\dots,a_n \in \mathcal{A}_{2n}$.

Proof. This is a direct consequence of the fact that $\widetilde{\Phi}$ is a Chevalley–Eilenberg cocycle. Indeed, starting from

$$0 = \tfrac{1}{n!}\,\epsilon_{b_1\dots b_n}\,(\delta\widetilde{\Phi})(a_{-1};a_0,a_1,\dots,a_n,y^{b_1},\dots,y^{b_n}) \tag{C.11}$$

where δ denotes the Chevalley–Eilenberg differential, one finds

$$\begin{aligned}0 &= \sum_{i=0}^{n}(-1)^i\,\mu([a_{-1},a_i]_*|a_0,\dots,\widehat{a}_i,\dots,a_n)\\ &\quad+\sum_{i<j}(-1)^{i+j}\,\mu(a_{-1};[a_i,a_j]_*,a_0,\dots,\widehat{a}_i,\dots,\widehat{a}_j,\dots,a_n)\\ &\quad+\tfrac{(-1)^{n-1}}{(n-1)!}\,\epsilon_{ab_1\dots b_{n-1}}\Big(\widetilde{\Phi}([a_{-1},y^a]_*;a_0,a_1\dots,a_n,y^{b_1},\dots,y^{b_{n-1}})\\ &\quad+\sum_{k=0}^{n}(-1)^k\,\widetilde{\Phi}(a_{-1};[a_k,y^a]_*,a_0,\dots,\widehat{a}_k,\dots,a_n,y^{b_1},\dots,y^{b_{n-1}})\Big)\,,\end{aligned} \tag{C.12}$$

[26] The map μ also vanishes identically if at least one of its n last arguments depends only on the y variables.

which, upon using $[-,y^a]_* = -\frac{\partial}{\partial p_a}$ and the fact that the latter can be factored out of the expression of $\widetilde{\Phi}$, reproduces (C.9). □

In plain words, μ is a Chevalley–Eilenberg cocycle, up to a total derivative in p. Moreover, this property implies that the extension of μ to forms on any manifold taking values in the Weyl algebra can be used to define a trace on the algebra of covariantly constant sections with respect to any given flat connection A.

Corollary C.3. *Let $\mathcal{X}$ be a smooth manifold, and A a flat connection on its Weyl bundle, associated to $T^*\mathcal{X}$. Then, for any pair of covariantly constant sections F,G with respect to A, the $\Omega(\mathcal{X})$-linear extension of μ verifies the cyclicity condition*

$$\mu([F,G]_*|A,\dots,A) \;\propto\; n\hbar\, d\mu(F|G,A,\dots,A) + \tfrac{\partial}{\partial p_a}\varphi_a(F|G,A,\dots,A)\,, \tag{C.13}$$

as well as the gauge invariance condition

$$\delta_\xi\mu(F|A,\dots,A) \;\propto\; n\, d\mu(F|\xi,A,\dots,A) + \tfrac{\partial}{\partial p_a}\varphi_a(F|\xi,A,\dots,A)\,, \tag{C.14}$$

for any gauge parameter ξ (i.e. any section of the Weyl bundle, not necessarily covariantly constant).

Proof. The proof is almost identical to that of [50, proposition 4.2]. For the sakes of completeness, let us repeat it.

First, let us use the almost-cocycle condition (C.9) verified by μ to write

$$\begin{aligned}\mu([F,G]_*|A,\dots,A) = n\Big(&\mu([F,A]_*|G,A,\dots,A) + \mu(F|[G,A]_*,A,\dots,A)\\ &- \tfrac{(n-1)}{2}\,\mu(F|[A,A]_*,G,A,\dots,A)\Big) + \tfrac{\partial}{\partial p_a}\varphi_a(F|G,A,\dots,A)\end{aligned} \tag{C.15}$$

where the dimension-dependent coefficients appear simply for combinatorial reason (several terms are identical since many of the arguments of μ are the same). Using the flatness of A and covariant constancy of F and G, we can replace all brackets by a differential term,

$$\begin{aligned}\mu([F,G]_*|A,\dots,A) = n\hbar\Big(&\mu(dF|G,A,\dots,A) + \mu(F|dG,A,\dots,A)\\ &+ (n-1)\,\mu(F|G,dA,\dots,A)\Big) + \tfrac{\partial}{\partial p_a}\varphi_a(F|G,A,\dots,A)\,,\end{aligned} \tag{C.16}$$

which reproduces the exact term in (C.13).

Similarly, the gauge variation

$$\delta_\xi\mu(F|A,\dots,A) = \tfrac{1}{\hbar}\,\mu([F,\xi]_*|A,\dots,A) + n\,\mu(F|d\xi + \tfrac{1}{\hbar}\,[A,\xi]_*,A,\dots,A) \tag{C.17}$$

can be recast as

$$\begin{aligned}\delta_\xi\mu(F|A,\dots,A) = n\Big(&\mu(dF|\xi,A,\dots,A) + \tfrac{1}{\hbar}\,\mu(F|[\xi,A]_*,A,\dots,A)\\ &+ (n-1)\,\mu(F|\xi,dA,\dots,A) + \mu(F|d\xi + \tfrac{1}{\hbar}\,[A,\xi]_*,A,\dots,A)\Big)\\ &+ \tfrac{\partial}{\partial p_a}\varphi_a(F|\xi,A,\dots,A)\,,\end{aligned} \tag{C.18}$$

upon using the almost-cocycle condition on the first term of equation (C.17), as well as the fact that A is flat and F covariantly constant. The two terms containing a commutator $[A,\xi]_*$ cancel one another, and the remaining terms add up to give (C.14).

□

Appendix D. Quantization of the cotangent bundle

D.1. From off-shell Segal to the quantization of the cotangent bundle

The off-shell Segal system presented in section 3.1 can be interpreted as a deformation quantization of the cotangent bundle $T^*\mathcal{X}$ over a given manifold $\mathcal{X}$, which we think of as spacetime here. Indeed, recall that the algebra of functions on the cotangent bundle which are polynomial in the momenta (fiber coordinates) is isomorphic to the algebra of symbols $\mathcal{S}(\mathcal{X})$ of differential operators on $\mathcal{X}$. Finding such an isomorphism (of vector spaces) between the space of symbols and the space of differential operators $\mathcal{D}(\mathcal{X})$ on $\mathcal{X}$,

$$\sigma : \mathcal{D}(\mathcal{X}) \xrightarrow{\sim} \mathcal{S}(\mathcal{X}) , \tag{D.1}$$

amounts to a quantization of the cotangent bundle (modulo some additional conditions, see e.g. [82] for more details and recent application in the context of HS gravity) in the sense that it allows one to define a star product on symbols, and hence on the algebra of polynomial functions on $T^*\mathcal{X}$, via the composition of differential operators, i.e.

$$\sigma(\widehat{D}_1) \star \sigma(\widehat{D}_2) = \sigma(\widehat{D}_1 \circ \widehat{D}_2) , \tag{D.2}$$

for any differential operators $\widehat{D}_1, \widehat{D}_2 \in \mathcal{D}(\mathcal{X})$. For instance, in the case of flat space $\mathcal{X} = \mathbb{R}^n$, this is nothing but an ordering prescription (e.g. Weyl or normal ordering). The connection 1-form A in the off-shell Segal system allows to define such an isomorphism as follows: assuming that A is of the form

$$A = dx^\mu \, e^a_\mu \, p_a + \cdots \tag{D.3}$$

where x^μ are coordinates on $\mathcal{X}$, the dots denote terms of higher orders in y and p, and e^a_μ is invertible, then there is a bijection between differential operators on $\mathcal{X}$ and sections of the bundle of Weyl algebra (in the variables (y^a, p_a) with $a = 1, \dots, \dim \mathcal{X} = n$) over the manifold $\mathcal{X}$, which are annihilated by

$$D := d + \tfrac{1}{\hbar} [A, -]_* , \tag{D.4}$$

where d denotes the de Rham differential on $\mathcal{X}$ and $*$ the fiberwise Moyal–Weyl product (meaning, between the (y^a, p_a) variables). Indeed, the equation $Df(x,p;y) = 0$, can be solved order by order in y (and $\hbar$) thanks to the fact that e^a_μ has an inverse (see appendix E for more details). In other words, the y-dependency of any function annihilated by D can be uniquely reconstructed, so that

$$Df = 0 \qquad \Leftrightarrow \qquad f = \tau(f_0) , \qquad f_0(x,p) = f(x,p;0) , \tag{D.5}$$

where τ is a bijection (whose inverse consists in setting $y = 0$). Now on the one hand, functions depending only on x^μ and p_a are nothing but polynomial functions on the cotangent bundle, expressed in a coordinate system wherein the tautological (or Liouville) 1-form reads $\vartheta = dx^\mu \, e^a_\mu \, p_a$. On the other hand, such functions identify with symbols of *fiberwise differential operators*, by which we mean differential operators in the y-variables, whose coefficients are smooth functions on $\mathcal{X}$ and formal series in y. As it turns out, Dolgushev showed that the space of such symbols, which are annihilated by the differential D, are in bijection with differential operators on $\mathcal{X}$ [98, theorem 3 and proposition 1]. In other words, the connection 1-form in the off-shell Segal system allows us to obtain a quantization of the cotangent bundle, by establishing an isomorphism between functions on $T^*\mathcal{X}$ (which are polynomial in momenta), and differential operators.

Considering that the cotangent bundle $T^*\mathcal{X}$ of any manifold $\mathcal{X}$ is symplectic, one can also quantize it using Fedosov's method [46]. As it turns out, the question of the precise relation between these two approaches was studied by Fedosov himself in [132]. The result is that, given a quantization of the cotangent bundle in the sense described in the previous paragraphs, one can build a Fedosov quantization of the cotangent bundle $T^*\mathcal{X}$ as follows.

- First, one should define a Fedosov connection on the Weyl algebra bundle over the symplectic manifold $T^*\mathcal{X}$. It therefore defines a differential on the space of forms valued in $\mathcal{A}_{2n}$ that should take the form

$$\widetilde{D} = \widetilde{d} + [\widetilde{A}, -]_*\,, \tag{D.6}$$

where $\widetilde{d}$ denotes the de Rham differential on $T^*\mathcal{X}$ and $\widetilde{A}$ is a $\mathcal{A}_{2n}$-valued connection 1-form on $T^*\mathcal{X}$, whose pieces linear in y and p read

$$\widetilde{A} = -d\pi_a\, y^a + dx^\mu\, e^a_\mu\, p_a + \cdots\,, \tag{D.7}$$

in a patch with coordinates (x^μ, π_a) of the cotangent bundle. Such a 1-form can be constructed from the 1-form in the parent Segal system, via

$$\widetilde{A}(x,\pi;y,p) = -d\pi_a\, y^a + A'\,, \qquad A' := A(x;y,p+\pi)\,. \tag{D.8}$$

The expression (D.6) does square to zero since

$$\widetilde{d}\widetilde{A} + \tfrac{1}{2}\,[\widetilde{A},\widetilde{A}]_* = 0 \qquad \Leftrightarrow \qquad \begin{cases} 0 = \big(\frac{\partial}{\partial\pi_a} - \frac{\partial}{\partial p_a}\big) A'_\mu\,, \\ 0 = dA' + \frac{1}{2}\,[A',A']_*\,. \end{cases} \tag{D.9}$$

The first equation is satisfied as a consequence of the particular dependency of A' in p and π, and the second one by virtue of the fact that both the action of the de Rham differential and the Moyal–Weyl star product commute with translation in p[27], and therefore this second equation follows from flatness of A. This is also consistent with the fact that the Fedosov connection $\widetilde{A}$ leads to a quantization of the cotangent bundle equipped with its canonical symplectic form $\widetilde{d}(e^a\,\pi_a)$. Indeed, adding the central term $-e^a\,\pi_a$ to $\widetilde{A}$ results in the minimal Fedosov connection whose curvature is $\widetilde{d}(e^a\,\pi_a)$, in accordance with the standard Fedosov quantization.

- Second, one should find covariantly constant sections of this differential $\widetilde{D}$, i.e. 0-forms $\widetilde{F}$ such that

$$\widetilde{D}\widetilde{F} = 0 \qquad \Leftrightarrow \qquad \begin{cases} 0 = \big(\frac{\partial}{\partial\pi_a} - \frac{\partial}{\partial p_a}\big)\,\widetilde{F}\,, \\ 0 = d\widetilde{F} + [A',\widetilde{F}]_*\,, \end{cases} \tag{D.10}$$

where the above equation has been split into two according to the natural basis $(dx^\mu, d\pi_a)$ of one-forms on $T^*\mathcal{X}$. The first equation is simplify solved by

$$\widetilde{F}(x,\pi;y,p) = F(x;y,p+\pi)\,, \tag{D.11}$$

and hence the second equation becomes equivalent to $DF = 0$.

[27] In fact, one can characterize the Moyal–Weyl star product as the unique star product on $\mathbb{R}^{2n}$ which is $\mathfrak{isp}(2n,\mathbb{R})$-equivariant [133].

In summary, the parent Segal system can be extended into a Fedosov differential and a covariantly constant section of the Weyl bundle over $T^*\mathcal{X}$.

D.2. Traces in Fedosov's quantization

As previously recalled, the basic principle of Fedosov's quantization is to establish an isomorphism, usually denoted τ, between functions on a symplectic manifold (M,ω) and sections of the Weyl bundle which are annihilated by the Fedosov differential $D_M := d_M + [A_M, -]_*$, where d_M denotes the de Rham differential on M, so as to define the star product $\star$ on $\mathcal{C}^\infty(M)$ as the pullback of the fiberwise Moyal–Weyl product $*$ of the corresponding sections of the Weyl bundle (under the action of τ). In other words,

$$\tau:\ \big(\mathcal{C}^\infty(M),\star\big) \xrightarrow{\ \sim\ } \big(H^0(D_M),*\big)\,, \tag{D.12}$$

is an isomorphism of associative algebras. This allows one to define a trace on $\big(\mathcal{C}^\infty(M),\star\big)$ by using structures defined on the Weyl bundle, namely the Fedosov differential and a Hochschild cocycle of the Weyl algebra. More precisely, given a function $f\in\mathcal{C}^\infty(M)$, its trace is given by

$$\mathrm{Tr}_{A_M}(f) := \int_M \Phi(\tau(f);A_M,\dots,A_M)\,, \tag{D.13}$$

where

$$\Phi:\underbrace{\mathcal{A}_{2n}\otimes\cdots\otimes\mathcal{A}_{2n}}_{2n\,\text{times}} \to \mathcal{A}_{2n}^*\,, \tag{D.14}$$

the representative of the Hochschild cohomology of $\mathcal{A}_{2n}$ with values in its dual $\mathcal{A}_{2n}^*$ derived in [50]. That this (multi-linear) map is a cocycle means that it verifies

$$0 = \Phi(f*a_0;a_1,\dots,a_{2n}) + \sum_{k=1}^{2n}(-1)^k\,\Phi(f;a_0,\dots,a_{k-1}*a_k,\dots,a_{2n}) - \Phi(a_{2n}*f;a_0,\dots,a_{2n-1})\,, \tag{D.15}$$

for any $a_0,\dots,a_{2n},f\in\mathcal{A}_{2n}$. Antisymmetrizing this identity in the a_i arguments yields[28],

$$0 = \sum_{\sigma\in\mathcal{S}_{2n+1}}(-1)^\sigma\Big(\Phi([f,a_{\sigma_0}]_*;a_{\sigma_1},\dots,a_{\sigma_{2n}}) - \tfrac{1}{2}\,\Phi(f;[a_{\sigma_0},a_{\sigma_1}]_*,\dots,a_{\sigma_{2n}})\Big)\,, \tag{D.16}$$

which can be re-written as

$$0 = \sum_{i=0}^{2n}(-1)^i\,\Phi([f,a_i]_*;a_0,\dots,\widehat{a}_i,\dots,a_{2n}) \tag{D.17}$$

$$\qquad + \sum_{i<j}(-1)^{i+j}\,\Phi(f;[a_i,a_j]_*,a_0,\dots,\widehat{a}_i,\dots,\widehat{a}_j,\dots,a_{2n})\,, \tag{D.18}$$

where hats denote the omission of the arguments, and with the understanding that the Φ appearing in this formula is the antisymmetric part in the last $2n$ arguments of the original cocycle. Using the above identity, and the fact that

$$d_M A_M + \tfrac{1}{2}[A_M,A_M]_* = 0\,, \qquad D_M F = 0 = D_M G\,, \tag{D.19}$$

[28] Note that this operation corresponds to producing a Chevalley–Eilenberg cocycle for the commutator algebra out of the original Hochschild cocycle.

for $F = \tau(f)$ and $G = \tau(g)$ the lifts of a pair of functions $f, g \in \mathcal{C}^\infty(M)$, one can check that [50, proposition 4.2]

$$\Phi\big([F,G]_*; A_M, \dots, A_M\big) = d_M\,\Phi\big(F; G, A_M, \dots, A_M\big)\,, \tag{D.20}$$

and

$$\delta_\xi\,\Phi(F; A_M, \dots, A_M) = d_M\,\Phi\big(F; \xi, A_M, \dots, A_M\big)\,, \tag{D.21}$$

for any section ξ of the Weyl bundle. In plain words, the obstruction for $\Phi(\tau(-); A_M, \dots, A_M)$ to vanish on $\star$-commutator and to be gauge-invariant is exact, which implies that for compactly supported functions, the operation (D.13) does define a trace, and that this definition is independent of the choice of Fedosov differential D_M.

Appendix E. Flat connection in the Weyl algebra

In this appendix, we give a proof of proposition 4.1, as well as some details about the construction of covariantly constant sections. In the rest of this section, $\deg(-)$ will refer to the Fedosov degree (4.2) on the Weyl algebra.

Proof of proposition 4.1. Let the Weyl bundle connection be given by $\mathcal{D} = d + \frac{1}{\hbar}[\varpi, -]_*$, where

$$\varpi = dx^\mu\, e_\mu^a\, p_a + \omega\,, \qquad \text{with} \qquad \omega := \sum_{n \geqslant 2} \varpi_{(n)}\,, \qquad \deg(\varpi_{(n)}) = n\,, \tag{E.1}$$

with e_μ^a invertible, and decompose its curvature

$$R := d\varpi + \tfrac{1}{2\hbar}[\varpi, \varpi]_* = \sum_{n \geqslant 1} R_{(n)}\,, \tag{E.2}$$

according to the Fedosov degree. Let us furthermore consider

$$A = \varpi + W\,, \qquad \text{with} \qquad W = \sum_{n \geqslant 2} W_{(n)}\,, \qquad \deg(W_{(n)}) = n\,, \tag{E.3}$$

whose curvature is given by

$$dA + \tfrac{1}{2\hbar}[A, A]_* = R - \delta W + \mathcal{D}_\omega W + \tfrac{1}{2\hbar}[W, W]_*\,, \tag{E.4}$$

where

$$\delta := -\tfrac{1}{\hbar}[e^a p_a, -]_* = e^a\,\tfrac{\partial}{\partial y^a}\,, \qquad \mathcal{D}_\omega := d + \tfrac{1}{\hbar}[\omega, -]_*\,, \tag{E.5}$$

are derivations of degree $\deg(\delta) = -1$ and $\deg(\mathcal{D}_\omega) \geqslant 0$ (by which we mean that $\mathcal{D}_\omega$ maps elements of degree k to elements of degree higher or equal to k). In fact, δ defines a differential on $\Omega(\mathcal{X}, \widehat{\mathcal{A}}_{2n})$ since $\delta^2 = 0$. At order n, the flatness condition for A reads

$$\delta W_{(n+1)} = R_{(n)} + (\mathcal{D}_\omega W)_{(n)} + \tfrac{1}{2\hbar}\sum_{k=2}^{n}[W_{(k)}, W_{(n+2-k)}]_*\,, \tag{E.6}$$

and in particular, the right hand side of the above equation only contains the components $W_{(k)}$ with $k \leqslant n$. Since the right hand side is δ-closed by virtue of the fact that $(-\delta + \mathcal{D}_\omega)R = 0$, which corresponds to the Bianchi identity of the connection $\mathcal{D}$, and $[W, [W, W]_*]_* = 0$ (Jacobi identity), the above equation can be read as the condition that the right hand side is δ-exact. In other words, assuming that there exists $W_{(k)}$ with $1 \leqslant k \leqslant n$ such that the curvature of A vanishes in degrees lower or equal to $n-1$, then the existence of $W_{(n+1)}$ ensuring the vanishing of the curvature of A up to degree n is encoded in the cohomology group of δ in (form) degree 1. The contracting homotopy h, introduced in (4.7), can therefore be used to solve $W_{(n+1)}$ in terms of the lower components, namely

$$W_{(n+1)} = h\left(R_{(n)} + (\mathcal{D}_\omega W)_{(n)} + \tfrac{1}{2\hbar}\sum_{k=2}^{n}[W_{(k)}, W_{(n+2-k)}]_*\right), \tag{E.7}$$

which solves (E.6) by virtue of the fact that $\{h,\delta\} = 1$ on $\mathcal{A}_{2n}$-valued 1-forms and $h^2 = 0$, and therefore uniquely specifies W in terms of the curvature of $\mathcal{D}$.

Having solved for A, we can now solve for F in a similar manner. Indeed, the covariant constancy condition reads

$$\delta F_{(n+1)} = (\mathcal{D}_\omega F)_{(n)} + \tfrac{1}{\hbar}\sum_{k=2}^{n+2}[W_{(k)}, F_{(n+2-k)}]_* = 0, \tag{E.8}$$

at order n. Here again, one can use the contracting homotopy to express $F_{(n+1)}$ in terms of lower order components, namely

$$F_{(n+1)} = h\left((\mathcal{D}_\omega F)_{(n)} + \tfrac{1}{\hbar}\sum_{k=2}^{n+2}[W_{(k)}, F_{(n+2-k)}]_*\right). \tag{E.9}$$

Note that $h(F) = 0$ due to the fact that F is a 0-form.

□

This way of constructing a flat connection starting from a possibly curved one is a variation on the Fedosov approach to the deformation quantization problem of symplectic manifold [46], as explained in [40, appendix A] (see also [98]). By construction, the completion of some possibly curved connection $\mathcal{D}$ into a flat connection A will only contain the curvature of $\mathcal{D}$, as well as covariant derivatives and contraction thereof.

Note that in the previous proof, we implicitly assumed that the connection $\mathcal{D}$ has no degree-0 piece, i.e. that ϖ does not contains a term $a \in \Omega^1(\mathcal{X})$, which lies in the center of the Weyl algebra. If such a term is present, proposition 4.1 still holds due to the following line of argument. Since a has degree 0, its field strength da contributes to the curvature of the connection $\varpi = a + e^a p_a + \omega$ in degree 0 as well and hence requires the introduction of a term $W_{(1)}$ of degree 1 in W to be compensated, i.e. $\delta W_{(1)} = da$. This equation is solved by $W_{(1)} = -e^a\,\partial_a a_b\, y^b$, and the obstruction to the flatness of the connection $\varpi + W_{(1)}$ is now of degree 2 and higher, and the degree 1 piece of this connection defines a new differential

$$\delta' := -\tfrac{1}{\hbar}\, e^a\,[p_a - \partial_a a_b\, y^b, -]_*, \tag{E.10}$$

with respect to which the condition that $A = \varpi + W$ is flat reads as in (E.6) upon replacing δ with δ'. Consequently, the existence of W also boils down to a cohomology problem, i.e. it is guaranteed provided that the cohomology of δ' is empty in form degree 1. As a matter of fact, one can show that the cohomology of δ' is empty in form degree greater or equal to one[29].

[29] Let us start by looking for δ'-cocycles, i.e. solutions to $\delta'\alpha = 0$. This equation can be expanded with respect to the degree in y as $\delta\alpha_{(k+1)} + \delta_a\alpha_{(k)} = 0$, where $\delta_a := \frac{1}{\hbar}[e^a\partial_a a_b\, y^b, -]_*$. Since δ and δ_a commute, $\delta_a\alpha_{(k)}$ is a δ-cocycle and hence the existence of $\alpha_{(k+1)}$ is ensured whenever α has form (strictly) positive form degree. This shows existence of δ'-cocycles in any positive form degree. Now given such a cocycle α, let us look for a solution to $\delta'\beta = \alpha$. Once again, expanding this equation with respect to the degree in y, one can see that it can be solved iteratively by $\beta_{(k+1)} = h(\alpha_{(k)} + \delta_a\beta_{(k)})$.

For the same reason, the lift of sections of $S(T\mathcal{X})$ to covariantly constant sections of the Weyl bundle $W(\mathcal{X})$ also exists.

Let us note that elements of the Weyl algebra which are linear in p form a Lie subalgebra under the star-commutator. In fact, the commutator of these elements reduces to the Poisson bracket, i.e.

$$\tfrac{1}{\hbar}\,[f,g]_* = \big(\tfrac{\partial f}{\partial y^a}\,\tfrac{\partial g}{\partial p_a} - \tfrac{\partial f}{\partial p_a}\,\tfrac{\partial g}{\partial y^a}\big) =: \{f,g\}\,, \tag{E.11}$$

for $f,g \in \mathcal{A}_{2n}$ linear in p[30]. Consequently, the completion of a connection ϖ which takes values in the subalgebra of elements linear in p will also share this property, since the recursive procedure (E.7) will only produce terms taking values in this subalgebra.

Let us also remark that, if ϖ is linear in p, and hence so is A, then the highest power in p contained in covariantly constant section F is the same as that of $F_{(0)}$. More precisely, if $F_{(0)}$ is an homogeneous polynomial in p of degree s, then F is a polynomial in p with terms of degrees s, $s-2$, $s-4$, … [40, proposition A.2].

Appendix F. Flattening the HS frame

Here we give an alternative procedure of encoding the unconstrained CHS fields into the flat connection A. In this context, it is convenient to employ an alternative degree:

$$\deg(y) = 1\,, \qquad \deg(p) = 0\,, \qquad \deg(\hbar) = 1\,, \tag{F.1}$$

and expand A as $A = A_0 + A_1 + \cdots$ into the homogeneous components. We treat $A_0 = \sum_s dx^\mu E_\mu^{b(s)} p_{b(s)}$ as the initial data and, as before, assume that the term linear in p in A_0 is invertible. In this setup, the role of δ is played by a new operator of degree -1:

$$\delta' = -\frac{1}{\hbar}[A_0,\cdot]_* = \delta + \sum_{l=1}^{\infty} \delta_l\,, \tag{F.2}$$

where δ_l has the total homogeneity degree l in p_a and $\hbar$. The standard homological algebra argument then shows that δ'-cohomology is empty in the nonvanishing form-degree. Moreover, the respective contracting homotopy operator can be chosen to have homogeneity 1 in y^a[31].

At degree 1, we take $A_1 = a_1 + \Gamma$, where $\Gamma = dx^\mu \omega_\mu{}^c{}_b\, y^b p_c$ encodes coefficients of the bare affine connection. The role of Γ is to maintain covariance of the procedure. Indeed, as we are going to see, with this choice only covariant derivatives of the fields entering $E_\mu(p)$ enter the construction. At degree zero, the equation $dA + \frac{1}{2\hbar}\,[A,A]_* = 0$ implies

$$\delta' a_1 = \nabla A_0 \qquad \Leftrightarrow \qquad dA_0 + \tfrac{1}{\hbar}\,[A_1,A_0]_* = 0 \tag{F.3}$$

and can be interpreted as the fact that the HS frame A_0 is covariantly constant with respect to HS connection A_1. Because A_0 is y-independent, $\delta' A_0 = 0$ so that the consistency condition $\delta'\nabla A_0 = 0$ is fulfilled and hence a_1 exists. Such a 'connection' A_1 can be considered the 'torsion-free' HS connection. Note that to avoid the dependency on the arbitrary connection Γ, one could take as Γ a Levi–Civita connection determined by the spin-2 frame e_μ^a and the constant Minkowski metric η^{ab}.

[30] This subalgebra is often referred to as the algebra of formal vector fields. This algebra is a central piece of 'formal geometry' [134, 135], and can be obtained as the prolongation (in the sense of Kobayashi [136]) of the inhomogeneous general linear algebra $\mathfrak{igl}(n,\mathbb{R})$.

[31] This happens because δ has no cohomology in nonvanishing form-degree, while the complex is quasi-isomorphic to $(H^\bullet(\delta),\Delta)$, where Δ is a differential induces by δ' in $H^\bullet(\delta)$. Note, however, that contracting homotopy implies inverting the HS frame and hence could result in elements which are nonpolynomial in p_a.

At degree 1, we have $\delta' A_2 = dA_1 + \frac{1}{2\hbar}[A_1, A_1]_*$, The consistency condition again holds:

$$\delta' \left[dA_1 + \frac{1}{2\hbar}[A_1, A_1]_* \right] = -d\delta' A_1 + \tfrac{1}{\hbar}[dA_0, A_1]_* + \tfrac{1}{\hbar}[\delta' A_1, A_1]_* = 0, \tag{F.4}$$

where we have made use of $\delta' A_1 = dA_0$, and hence A_2 also exists.

One then proceeds by the standard induction. Let us recall how it goes. Introducing $\Omega = \hbar d + A$, the zero-curvature condition is equivalent to $[\Omega, \Omega]_* = 0$. At degree k, the equation reads

$$\delta' A_{k+1} + \frac{1}{2\hbar}[\Omega^k, \Omega^k]_*|_k = 0, \qquad \text{with} \qquad \Omega^k := \hbar d + \sum_{l=0}^{k} A_l, \tag{F.5}$$

where $C|_k$ denotes the degree k component of C, and it is assumed that the equation is already solved to order k. The consistency condition $\delta'([\Omega^k, \Omega^k]_*|_k) = 0$ is the degree k component of the identity $[\Omega^k, [\Omega^k, \Omega^k]_*]_* = 0$, where the induction assumption has been used. It is therefore satisfied, which implies the existence of A_{k+1}. In other words, any $A_0 = dx^\mu E_\mu(p)$ has a unique (modulo gauge transformations) lift to A satisfying the flatness condition $dA + \frac{1}{2\hbar}[A, A]_* = 0$. It is natural to call $dx^\mu E_\mu(p)$ a HS frame-field.

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